SF 1684 Algebra and Geometry
Lecture 5

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Topics for Today

1. Matrices as Vector Space: Addition and Scalar Multiplication
2. Multiplying Matrices by Vectors
3. Multiplying Two Matrices

Row and Column Vectors

Recall we say that

m columns

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \quad n \text{ rows}$$

is an $m \times n$ matrix.

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is an $m \times n$ matrix.

We will denote

$$\vec{r}_i := \begin{pmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \end{pmatrix}$$

as the *i*-**th row vector** of A and consider it as a "$1 \times n$ matrix".

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1st column vector

nth column Vector

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as the i-**th row vector** of A and consider it as a "$1 \times n$ matrix". Further, we will denote

$$\vec{c}_j := \begin{pmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{m,j} \end{pmatrix}$$

as the j-**th column vector** of A and consider it as a "$m \times 1$ matrix".

Shorthand Notations

We can then created three differing shorthand notations for the matrix A:

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To save space, and if it can be inferred from context, we will write just

$$(a_{i,j})_{i,j} \text{ or } (a_{i,j}) \qquad \text{instead of} \qquad (a_{i,j})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$$

Matrices as Vector Spaces

Theorem

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1. *The* **zero matrix** *being*

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

2. *Addition being done "coordinate-wise"* component -wise

$$(a_{i,j})_{i,j} + (b_{i,j})_{i,j} = (a_{i,j} + b_{i,j})_{i,j}$$

3. *Scalar multiplication also being done "coordinate-wise"* component-wise

$$c(a_{i,j})_{i,j} = (ca_{i,j})_{i,j}$$

Examples

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divide by 2

multiply by $\frac{1}{2}$

Examples

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CAUTION!!!!!!

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$$\overset{2\times 3}{A = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 5 & 11 \end{pmatrix}} \qquad \overset{2\times 3}{B = \begin{pmatrix} 5 & -2 & 0 \\ -9 & 2 & 2 \end{pmatrix}}$$

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CAUTION!!!!!!

You can only add matrices of the *same* dimension!

Examples

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CAUTION!!!!!!

You can only add matrices of the *same* dimension! That is, if

$$C = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 5 & -2 \\ 7 & 11 & 21 \end{pmatrix}$$

then it does not even make sense to consider things like $A + C$ or $2C + 5B$!

Multiplication of Matrices

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$$A * B = (a_{i,j})_{i,j} * (b_{i,j})_{i,j} = (a_{i,j} b_{i,j})_{i,j}$$

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This is not good!!

However, we must remember that we are interested in matrices in relation to solving systems of linear equations.

As it turns out defining the multiplication of matrices in this way does not help us understand this.

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Therefore, there is no real way to multiply them except for the naive way.

That is, if $A = \begin{pmatrix} a \end{pmatrix}$ and $X = \begin{pmatrix} x \end{pmatrix}$ it must be that

$$A * X = \begin{pmatrix} ax \end{pmatrix}$$

Multiplying 1×1 Matrices 2

However, since $X = \begin{pmatrix} x \end{pmatrix}$ is essentially just an element of $\mathbb{R} = \mathbb{R}^1$, we can view it as vector: $X = \vec{x} = \begin{bmatrix} x \end{bmatrix}$.

Multiplying 1×1 Matrices 2

However, since $X = \left(x\right)$ is essentially just an element of $\mathbb{R} = \mathbb{R}^1$, we can view it as vector: $X = \vec{x} = \left[x\right]$.

Thus we can define how to multiply a 1×1 matrix $A = \left(a\right)$ with a vector in $\mathbb{R}^1$, $\vec{x} = \left[x\right]$:

$$A * \vec{x} = \left[ax\right] = (ax)$$

$\mathbb{R}^1$

1×1

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$$A * \vec{x} = \left[ax\right]$$

We want to relate this back to solving linear equations.

$$(A \mid \vec{b}) \longrightarrow \underbrace{\bar{a}\,\bar{x}}_{A * \vec{x}} = b$$

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We want to relate this back to solving linear equations. So if $\vec{b} = \begin{bmatrix} b \end{bmatrix}$, is a vector in $\mathbb{R}^1$, then we see that $\vec{x} = \begin{bmatrix} x \end{bmatrix}$ solves to the 1×2 augmented matrix $(A|\vec{b}) = (a|b)$ if and only if $ax = b$.

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Conclusion

If A is a 1×1 matrix and $\vec{b}$ is a vector in $\mathbb{R}^1$, then a vector $\vec{x}$ in $\mathbb{R}^1$ solves the augmented matrix $(A|\vec{b})$ if and only

$$A * \vec{x} = \vec{b}$$

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Want

For any matrix A and any vector $\vec{x}$, we want to define $A * \vec{x}$ such that $\vec{x}$ solves the augmented matrix $(A|\vec{b})$ if and only if $A * \vec{x} = \vec{b}$.

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#columns = #variables CAUTION!!!!!!

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If A is an $m \times n$ matrix then it corresponds to a system of linear equations in n variables. Thus, in order for $\vec{x}$ to solve the augmented matrix $(A|\vec{b})$, $\vec{x}$ *must* be in $\mathbb{R}^n$!

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So, we would NOT be able to define the multiplication of a 3×2 matrix by a vector in $\mathbb{R}^5$ in this way.

Multiplying Matrices by Vectors 2

So, given an augmented matrix

$$(A|\vec{b}) := \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} & b_m \end{array}\right)$$

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we recall that a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ solves the augmented matrix if

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something involving A & $\vec{x}$

$$A * \vec{x} := \begin{bmatrix} \overbrace{a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n} \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \dots a_{m,n}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \vec{b}$$

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \dots a_{m,n}x_n &= b_m \end{aligned}$$

Multiplying Matrices by Vectors Definition

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Given an $m \times n$ matrix and a vector in $\mathbb{R}^n$

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we **define** $A * \vec{x}$ to be

$$A * \vec{x} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots a_{m,n}x_n \end{bmatrix}$$

Multiplying Matrices by Vectors Theorem

Theorem

Given an $m \times n$ matrix A and a vector $\vec{b}$ in $\mathbb{R}^m$, then a vector $\vec{x}$ in $\mathbb{R}^n$ solves the augmented matrix $(A|b)$ if and only if

$$A * \vec{x} = \vec{b}$$

Proof.

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Proof.

By construction. $\square$

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Proof.

By construction. □

Notation: from now on we will just write $A\vec{x}$ instead of $A * \vec{x}$ to indicate the multiplication of a matrix by a vector in this way.

Exercise

Calculate $A\vec{x}$ in the following and interpret the result in terms of a solution to a system of linear equations

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 5 & 10 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\times1+(-1)\times2 \\ 2\times3+(-1)\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Not possible!

dimension don't match

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \underset{\vec{b}}{\underset{\|}{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}} \Longrightarrow \underset{\begin{bmatrix} x \\ y \end{bmatrix}}{\underset{\|}{\vec{x}}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ solves the SOLE}$$

$$\left.\begin{aligned} x+2y &= 0 \\ 3x+4y &= 2 \end{aligned}\right\} \begin{matrix} b_1 \\ b_2 \end{matrix}$$

Extra Work Space

2) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$A\bar{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \times 1 + 0 \times 0 + 1 \times 1 \\ -1 \times 0 + 0 \times 1 + 0 \times 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{b}} \begin{matrix} b_1 \\ b_2 \end{matrix}$$

$\Rightarrow \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ Solves SoLE.

$$\begin{aligned} x + z &= 0 = b_1 \\ y &= 0 = b_2 \end{aligned}$$

3) $A = \begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 5 & 10 \end{bmatrix}$

Suggest $B = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 1 & 0 \\ 5 & 10 & 0 \end{bmatrix}$ ← a different matrix

Not A! →

Remarks

1. If A is an $m \times n$ matrix then $A\vec{x}$ only makes sense if $\vec{x}$ is in $\mathbb{R}^n$

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1. If A is an $m \times n$ matrix then $A\vec{x}$ only makes sense if $\vec{x}$ is in $\mathbb{R}^n$.
2. If A is an $m \times n$ matrix and $\vec{x}$ is in $\mathbb{R}^n$ then $A\vec{x}$ is a vector in $\mathbb{R}^m$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} \text{ in } \mathbb{R}^3$$

3×2 matrix $\qquad \mathbb{R}^2$

Remarks

1. If A is an $m \times n$ matrix then $A\vec{x}$ only makes sense if $\vec{x}$ is in $\mathbb{R}^n$.
2. If A is an $m \times n$ matrix and $\vec{x}$ is in $\mathbb{R}^n$ then $A\vec{x}$ is a vector in $\mathbb{R}^m$.
3. We may have $A\vec{x} = \vec{0}$ even if A is not the zero matrix and $\vec{x} \neq \vec{0}$.

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1. If A is an $m \times n$ matrix then $A\vec{x}$ only makes sense if $\vec{x}$ is in $\mathbb{R}^n$.
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3. We may have $A\vec{x} = \vec{0}$ even if A is not the zero matrix and $\vec{x} \neq \vec{0}$.

Theorem

$\vec{x}$ is a homogeneous solution to A if and only if $A\vec{x} = \vec{0}$.

$\vec{x}$ is homo if it solves $(A|\vec{0}) \Leftrightarrow A\vec{x} = \vec{0}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\vec{b} \quad\quad\quad\quad\quad\quad \vec{b}$

Multiplying Matrices by Vectors: Row Vectors Dot Product

Theorem

Recall that if $A = (a_{i,j})$ *is an* $m \times n$ *matrix then the row vectors,* $\vec{r}_i = \begin{pmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \end{pmatrix}$, *are vectors in* $\mathbb{R}^n$.

Multiplying Matrices by Vectors: Row Vectors Dot Product

Theorem

Recall that if $A = (a_{i,j})$ is an $m \times n$ matrix then the row vectors, $\vec{r}_i = \begin{pmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \end{pmatrix}$, are vectors in $\mathbb{R}^n$. Then for any vectors $\vec{x}$ in $\mathbb{R}^n$, we get

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

$\vec{r}_1 \cdot \vec{x} \in \mathbb{R}$

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \end{bmatrix}$$

$$\vec{r}_1 \cdot \vec{x} = (a_{11} \quad a_{12} \quad \cdots \quad a_{1n}) \cdot (x_1, x_2 \quad \cdots \quad x_n) = a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n$$

Linearity Properties

Theorem (Linearity Properties)

If A is an $m \times n$ matrix, $\vec{x}$ and $\vec{y}$ vectors in $\mathbb{R}^n$ and c a scalar then

1. $A(c\vec{x}) = cA\vec{x}$
2. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ $\longrightarrow$ $A(x+y) = Ax + Ay$

1) $A = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$ $\quad A \cdot (c\vec{x}) = \begin{pmatrix} r_1 \cdot (c\vec{x}) \\ \vdots \\ r_m \cdot (c\vec{x}) \end{pmatrix} = \begin{pmatrix} c\ r_1 \cdot x \\ \vdots \\ c\ r_m \cdot x \end{pmatrix} = c \begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix}$

$$A(\vec{x}+\vec{y}) = \begin{pmatrix} r_1 \cdot (x+y) \\ \vdots \\ r_m (x+y) \end{pmatrix} = \begin{pmatrix} r_1 \cdot x + r_1 \cdot y \\ \vdots \\ r_m \cdot x + r_m \cdot y \end{pmatrix} = \begin{pmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{pmatrix} + \begin{pmatrix} r_1 \cdot y \\ \vdots \\ r_m \cdot y \end{pmatrix}$$

$$= A\vec{x} + A \cdot \vec{y}$$

Homogeneous Solutions are a Vector Space: Reproof

Theorem

Given a matrix A, the set of homogeneous solutions of A form a vector space. Equivalently, the following holds

1. $\vec{0}$ *is a homogeneous solution*
2. *If $\vec{x}$ is a homogeneous solution and $c \in \mathbb{R}$ then $c\vec{x}$ is also a homogeneous solution*
3. *If $\vec{x}$ and $\vec{y}$ are homogeneous solutions than so is $\vec{x} + \vec{y}$*

① $A\vec{0} = \begin{bmatrix} r_1 \cdot \vec{0} \\ \vdots \\ r_m \cdot \vec{0} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

② $0 = A\vec{x} \rightarrow A(c\vec{x}) = c \cdot A\vec{x} = c \cdot 0 = 0$

③ $A\vec{x} = 0$, $A\vec{y} = 0$ $\Rightarrow A(x+y) = Ax + Ay = 0 + 0 = 0$

Multiplication of Matrices

Suppose A is an $m \times n$ matrix and B is a $n \times \ell$ matrix.

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$$B = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_\ell \end{pmatrix}$$

where the $\vec{c}_i$ are the column vectors of B and so are vectors in $\mathbb{R}^n$.

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$A\vec{c}_i$ makes sense
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where the $\vec{c}_i$ are the column vectors of B and so are vectors in $\mathbb{R}^n$. Hence, we know how to multiply each $\vec{c}_i$ by A. We now can think of multiplying B by A by distributing it:

Definition

Let A be an $m \times n$ matrix and B be a $n \times \ell$ matrix with column vectors $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_\ell$, then we define

$$AB = A \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_\ell \end{pmatrix} = \begin{pmatrix} A\vec{c}_1 & A\vec{c}_2 & \dots & A\vec{c}_\ell \end{pmatrix}$$

$m \times \ell$ matrix

$\mathbb{R}^m$ $\mathbb{R}^m$ $\mathbb{R}^m$

Remarks on Multiplication of Matrices

$$AB = A\begin{pmatrix}\vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_\ell\end{pmatrix} = \begin{pmatrix}A\vec{c}_1 & A\vec{c}_2 & \dots & A\vec{c}_\ell\end{pmatrix}$$

Remarks on Multiplication of Matrices

$$AB = A\begin{pmatrix}\vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_\ell\end{pmatrix} = \begin{pmatrix}A\vec{c}_1 & A\vec{c}_2 & \dots & A\vec{c}_\ell\end{pmatrix}$$

1. If A is a $m \times n$ matrix and B is a $k \times \ell$ matrix then the $\vec{c}_i$ are in $\mathbb{R}^k$ and hence AB only makes sense if $n = k$ (i.e. the number of columns of A must be equal to the number of rows of B).

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2. It is very possible that AB makes sense while BA does not even make sense!

$n = k$ AB make

$m \neq \ell$ BA does not make

$k \times \ell$ $m \times n$

Remarks on Multiplication of Matrices

$$AB = A\begin{pmatrix}\vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_\ell\end{pmatrix} = \begin{pmatrix}A\vec{c}_1 & A\vec{c}_2 & \dots & A\vec{c}_\ell\end{pmatrix}$$

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2. It is very possible that AB makes sense while BA does not even make sense!
3. Since $\vec{c}_i$ are vectors in $\mathbb{R}^n$, the $A\vec{c}_i$ are vectors in $\mathbb{R}^m$ and so AB is an $m \times \ell$ matrix.

$$(\underline{m} \times n) * (n \times \underline{\ell}) \implies \underline{m} \times \underline{\ell}$$

Exercise

Exercise

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Computer AB, AC, BA, BC, CA and CB or state that they don't make sense

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2\times 1 + 1\times -1 & 0\times 2 + 2\times 1 & 1\times 2 + 1\times 1 \\ 1\times 1 + 0\times -1 & 0\times 1 + 2\times 0 & 1\times 1 + 0\times 1 \\ 1\times 0 + 0\times -1 & 0\times 0 + 0\times 0 & 1\times 0 + 0\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = AB$$

3×2 $\quad$ 2×3 $\rightarrow$ 3×3

Extra Work Space

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2\times 1 + 0\times 1 + 1\times 0 & 1\times 1 + 0\times 0 + 1\times 0 \\ 2\times -1 + 2\times 1 + 0\times 0 & 1\times(-1) + 2\times 0 + 1\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = BA$$

$$\boxed{AB \neq BA}$$

Multiplying Matrices: Row Vector, Column Vector Dot Product

Theorem

Let A be an $m \times n$ matrix with row vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m$ and B be an $n \times \ell$ matrix with column vectors $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_\ell$. Then

$$AB = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{pmatrix} \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_\ell \end{pmatrix} = (\vec{r}_i \cdot \vec{c}_j)_{\substack{i=1,\dots,m \\ j=1,\dots,\ell}}.$$

row of A

Column of B

That is, the (i, j)-th entry of the $m \times \ell$ matrix AB is $\vec{r}_i \cdot \vec{c}_j$.

$$AB = \begin{pmatrix} Ac_1 & Ac_2 & \cdots & Ac_\ell \end{pmatrix} = \left(\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} c_1 \quad \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} c_2 \; \cdots \; \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} c_\ell \right) = \begin{bmatrix} r_1 \cdot c_1 & \cdots & r_1 \cdot c_\ell \\ \vdots & \ddots & \vdots \\ r_m \cdot c_1 & \cdots & r_m \cdot c_\ell \end{bmatrix}$$

Multiplying Matrices: Row Vector, Column Vector Dot Product

Theorem

Let A be an $m \times n$ matrix with row vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ and B be an $n \times \ell$ matrix with column vectors $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_\ell$. Then

$$AB = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{pmatrix} \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \ldots & \vec{c}_\ell \end{pmatrix} = (\vec{r}_i \cdot \vec{c}_j)_{\substack{i=1,\ldots,m \\ j=1,\ldots,\ell}}.$$

That is, the (i,j)-th entry of the $m \times \ell$ matrix AB is $\vec{r}_i \cdot \vec{c}_j$.

Proof.

Follows from the similar theorem about matrices multiplied by vectors and the definition of matrix multiplication.

□

Definition

We call an $n \times n$ matrix $D = (d_{i,j})$ **diagonal** if $d_{i,j} = 0$ whenever $i \neq j$:

$$D = \begin{pmatrix} d_{1,1} & 0 & \dots & 0 \\ 0 & d_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{n,n} \end{pmatrix}$$

Multiplying a Matrix by a Diagonal Matrix on the Right

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

and A is an $m \times n$ matrix with entries $a_{i,j}$,

Multiplying a Matrix by a Diagonal Matrix on the Right

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

not DA
doesn't make
sense unless
$\underline{m = n}$

and A is an $m \times n$ matrix with entries $a_{i,j}$, then

$$AD = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

Multiplying a Matrix by a Diagonal Matrix on the Right

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

and A is an $m \times n$ matrix with entries $a_{i,j}$, then

$$AD = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

$$= \begin{pmatrix} d_1 a_{1,1} & d_2 a_{1,2} & \dots & d_n a_{1,n} \\ d_1 a_{2,1} & d_2 a_{2,2} & \dots & d_n a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ d_1 a_{m,1} & d_2 a_{m,2} & \dots & d_n a_{m,n} \end{pmatrix}$$

Multiplying a Matrix by a Diagonal Matrix on the Left

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \end{pmatrix}$$

and A is an $n \times m$ matrix with entries $a_{i,j}$,

Multiplying a Matrix by a Diagonal Matrix on the Left

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \end{pmatrix}$$

and A is an $n \times m$ matrix with entries $a_{i,j}$, then

$$DA = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$$

Multiplying a Matrix by a Diagonal Matrix on the Left

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \end{pmatrix}$$

and A is an $n \times m$ matrix with entries $a_{i,j}$, then

$$DA = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$$

$$= \begin{pmatrix} d_1 a_{1,1} & d_1 a_{1,2} & \dots & d_1 a_{1,m} \\ d_2 a_{2,1} & d_2 a_{2,2} & \dots & d_2 a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ d_n a_{n,1} & d_n a_{n,2} & \dots & d_n a_{n,m} \end{pmatrix}$$

multiply 1st row by d_1 2nd row by d_2 and so on.

Multiplying Two Diagonal Matrices

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n \end{pmatrix}$$

Multiplying Two Diagonal Matrices

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n \end{pmatrix}$$

then

$$DE = \begin{pmatrix} d_1 e_1 & 0 & \dots & 0 \\ 0 & d_2 e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n e_n \end{pmatrix}$$

Multiplying Two Diagonal Matrices

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n \end{pmatrix}$$

then

$$DE = \begin{pmatrix} d_1 e_1 & 0 & \dots & 0 \\ 0 & d_2 e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n e_n \end{pmatrix} = \begin{pmatrix} e_1 d_1 & 0 & \dots & 0 \\ 0 & e_2 d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n d_n \end{pmatrix}$$

Multiplying Two Diagonal Matrices

Let

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n \end{pmatrix}$$

then

$$DE = \begin{pmatrix} d_1 e_1 & 0 & \dots & 0 \\ 0 & d_2 e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n e_n \end{pmatrix} = \begin{pmatrix} e_1 d_1 & 0 & \dots & 0 \\ 0 & e_2 d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n d_n \end{pmatrix} = ED$$

$DE = ED$ but this is a very special case