SF 1684 Algebra and Geometry
Lecture 4

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Topics for Today

1. Solving Augmented Matrices
2. Reduced Row Echcelon Form

Last Class

1. Showed that solving a system of linear equations is equivalent to finding a solution to an augmented matrix.
2. Showed that this can be done using *equation operation* on the equations or *row operations* on the rows of the augmented matrix.

Ideal Situation

Ideally, for a system of linear equations we would want to perform equation operations to reduce it

$$\left(\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m \end{aligned}\right) \implies \begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ &\vdots \\ x_n &= c_n \end{aligned}$$

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For matrices this would correspond to performing row operations to reduce it

$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} & b_m \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_n \end{array}\right)$$

Non-Ideal Situation

However, the ideal situation does not always happen...

Exercise

Find all solutions to the system of linear equations

$$x + 4y + z = 2 \qquad \text{is a line.}$$

$$2x + 3z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 2 & 0 & 3 & 2 \end{array}\right] \quad R_2 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 0 & -8 & 1 & -2 \end{array}\right] \quad -\tfrac{1}{8} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 0 & 1 & -\frac{1}{8} & \frac{1}{4} \end{array}\right] \quad R_1 - 4R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 1 \\ 0 & 1 & -\frac{1}{8} & \frac{1}{4} \end{array}\right]$$

More Work Space

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 1 \\ 0 & 1 & -\frac{1}{8} & 4 \end{array}\right] \\ \begin{array}{cccc} x & y & z & t \end{array} \end{array} \longrightarrow \begin{array}{l} x + \frac{3}{2} z = 1 \\ y - \frac{1}{8} z = 4 \end{array}$$

Ex: $z = 8$: $\begin{array}{l} x + 12 = 1 \\ y - 1 = 4 \end{array} \longrightarrow \begin{array}{l} x = -11 \\ y = 5 \end{array}$

Solution: $(x, y, z) = (-11, 5, 8)$

Let $z = t$, a parameter,

$$\begin{array}{l} x + \frac{3}{2} t = 1 \\ y - \frac{1}{8} t = 4 \\ z = t \end{array} \longrightarrow \begin{array}{l} x = 1 - \frac{3}{2} t \\ y = 4 + \frac{1}{8} t \\ z = t \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - \frac{3}{2} t \\ 4 + \frac{1}{8} t \\ 0 + t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3/2 \\ 1/8 \\ 1 \end{bmatrix} t, \quad \text{for any } t \in \mathbb{R}$$

parametric equation of a line.

Geometric Reasoning for Our Solution

We know that both formulas $\underbrace{x + 4y + z = 2}$ and $\underbrace{2x + 3z = 2}$ describe a *plane* in $\mathbb{R}^3$.

point-normal formulas

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$

planes

Geometric Reasoning for Our Solution

We know that both formulas $x + 4y + z = 2$ and $2x + 3z = 2$ describe a *plane* in $\mathbb{R}^3$. Hence, finding the set of points (x, y, z) that satisfy both equations is the same as finding the set of points that are on both planes,

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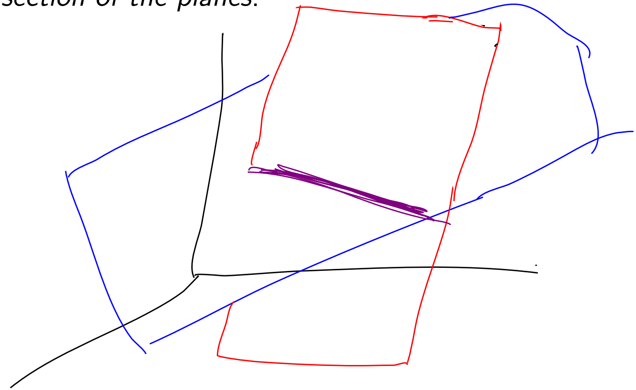

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Reducing Matrices

We note that in the previous example, we reduced the augmented matrix as much as possible

$$\underline{\left(\begin{array}{ccc|c} 1 & 4 & 1 & 2 \\ 2 & 0 & 3 & 2 \end{array}\right)} \implies \underline{\left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & -\frac{1}{8} & \frac{1}{4} \end{array}\right)}$$

good

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This is an example of **reduced row echelon form**.

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$$\begin{bmatrix} 0 & 0 & 0 & \cdots & & & 0 & \\ 0 & 0 & 0 & 1 & 2 & 5 & 6 & \cdots \\ 1 & 3 & 5 & 6 & 7 & & & \end{bmatrix}$$

leading 1s

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$$\begin{bmatrix} 0 & 1 & - & - & - & & \\ 0 & 0 & \cdots & 1 & - & \cdots & - \\ 0 & - & \cdots & & & & 0 \\ 0 & 0 & - & \cdots & & & 0 \\ 0 & 0 & - & - & - & & 0 \end{bmatrix}$$

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3. In any two successive rows that do not consist entirely of zeroes, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

$$\begin{bmatrix} 0 & 0 & 1 & - & - & - & - & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & - & - & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & - & - & - & \cdot & & & & 0 \end{bmatrix}$$

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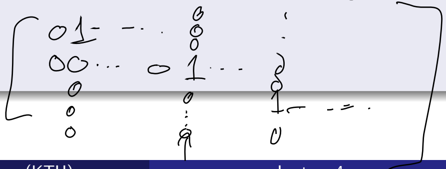
3. In any two successive rows that do not consist entirely of zeroes, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zero everywhere else.



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If the first three properties hold, we say the matrix is in **Row Echelon Form** (REF).

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If the first three properties hold, we say the matrix is in **Row Echelon Form** (REF).

The process of transforming a matrix into RREF is often called "reducing".

Examples of RREF and REF

Reduced Row Echelon Form:

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Row Echelon Form:

not RREF

$$\begin{pmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Neither form:

$$\begin{pmatrix} 0 & 1 & 0 & 4 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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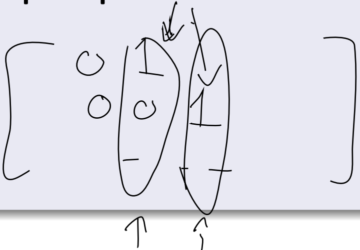
Definition

1. The positions that have a leading 1 in REF or RREF are sometimes referred to as **pivot positions** and their columns as **pivot columns**.



pivot column

Definitions and Terminology

Definition

1. The positions that have a leading 1 in REF or RREF are sometimes referred to as **pivot positions** and their columns as **pivot columns**.
2. The **rank** of a matrix A is the number of leading 1s when it is reduced to REF or RREF. We denote this as $\mathrm{rk}(A)$.

$$A = \begin{bmatrix} 0 & 0 & 1 & \cdot & - & & \\ 0 & 0 & 0 & 0 & 1 & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \qquad \mathrm{rk}(A) = 2$$

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3. The parameter t we saw in the example is referred to as a **free variable**. They correspond to columns that aren't pivot columns. Note, it is possible to have multiple free variables!

$$\begin{array}{ccc} x & y & z \\ \hline \end{array}$$
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

pivot columns   free variable

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Facts

Fact

1. $\#$ *columns* $= \#$ *variables* $= rk(A) + \#$ *free variables*

rk = # of pivot columns

free variables = # of non-pivot columns

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1. $\#$ *columns* $= \#$ *variables* $= rk(A) + \#$ *free variables*
2. *If* $rk(A) = \#$ *variables, then there is a unique solution to A*
   homogeneous

If $rk(A) = \#$ variables then we have

$= \#$ columns

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & \ddots & & 0 \\ 0 & 0 & & 1 & \vdots \\ 0 & \cdots & & 0 & \\ 0 & \cdots & & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_1 = s_1 \\ x_2 = s_2 \quad \cdots \end{array}$$

Facts

Fact

1. $\#$ columns $= \#$ variables $= rk(A) + \#$ free variables
2. If $rk(A) = \#$ variables, then there is a unique homogeneous solution to A
3. If $rk(A) < \#$ variables, then there are infinitely many homogeneous solutions to A.

If $rk(A) < \#$ variables then we have a free variable and so the solutions will be something like $t\vec{v}$

t can be any real number.

Solving Augmented Matrices in REF and RREF

Exercise

Find all the solutions to following augmented matrices

$$(A|\vec{a}) = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \vec{a} \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & 4 & 0 & -5 & 7 \\ 0 & 0 & 0 & 1 & 0 & -3 \end{array}\right) \end{array} \qquad (B|\vec{b}) = \left(\begin{array}{cccc|c} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivot columns

free variables

$$(C|\vec{c}) = \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 1 \end{array}\right) \end{array} \longrightarrow \begin{array}{l} x = 2 \\ 0x + 0y = 1 \end{array}$$

$$\begin{array}{l} x_1 + 2x_3 + 2x_5 = 3 \\ x_2 + 4x_3 - 5x_5 = 7 \\ x_4 = -3 \end{array} \longrightarrow$$

let $x_3 = t$ $\quad x_5 = s$

$$\begin{aligned} x_1 &= 3 - 2t - 2s \\ x_2 &= 7 - 4t + 5s \\ x_3 &= t \\ x_4 &= -3 \\ x_5 &= s \end{aligned}$$

$0 = 1$

no solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2-2t-2s \\ 7-4t+5s \\ t \\ -3 \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix} s$$

plane in $\mathbb{R}^5$.

$$\begin{array}{cccc} & & & \\ \end{array}\left[\begin{array}{cccc|c} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$\longrightarrow$

$$x_2 + 2x_3 + 6x_4 = 0$$
$$x_3 - x_4 = 0$$
$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

$x_1 = s \quad x_4 = t$

$$x_1 = s$$
$$x_2 = -2x_3 - 6t$$
$$x_3 = t$$
$$x_4 = t$$

$\longrightarrow$

$$x_1 = s$$
$$x_2 = -2t - 6t = -8t$$
$$x_3 = t$$
$$x_4 = t$$

Plane in $\mathbb{R}^4$!

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ -8 \\ 1 \\ 1 \end{bmatrix} t$$

can ignore this

Consistent

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Theorem

If an augmented matrix $(A|\vec{b})$ *has a row of the form*

$$\left(\begin{array}{cccc|c} 0 & 0 & \dots & 0 & c \end{array}\right)$$

when brought to RREF,

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If an augmented matrix $(A|\vec{b})$ has a row of the form

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when brought to RREF, then $(A|\vec{b})$ is consistent if $c = 0$

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Theorem

If an augmented matrix $(A|\vec{b})$ *has a row of the form*

$$\begin{pmatrix} 0 & 0 & \dots & 0 & \big| & c \end{pmatrix} \longrightarrow 0x_1 + 0x_2 + \dots = c$$

when brought to RREF, then $(A|\vec{b})$ *is consistent if* $c = 0$*and inconsistent if* $c \neq 0$.

Exercise

Exercise

For which values of a is the following system consistent:

$$x_1 + x_2 + \cancel{2x_3} + 4x_4 = 1$$

$$2x_1 + 4x_2 + x_4 = 1$$

$$x_1 - x_2 + 11x_4 = a$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 1 \\ 2 & 4 & 0 & 1 & 1 \\ 1 & -1 & 0 & 11 & a \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 1 \\ 0 & 2 & 0 & -7 & -1 \\ 0 & -2 & 0 & 7 & a-1 \end{array}\right] R_3 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 1 \\ 0 & 2 & 0 & -7 & -1 \\ 0 & 0 & 0 & 0 & a-2 \end{array}\right]$$

consistent only if $a-2=0$

only if $\underline{a=2}$

Extra Work Space

Row Equivalence

Definition

Two matrices are **row equivalent** if there is a sequence of row operations that transforms one into the other.

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Theorem

For any matrix A, there exists a unique matrix S that is in RREF that is row equivalent to A.

Proof by Gauss-Jordan Elimination

1. Locate the leftmost column that does not consist entirely of zeros

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ & 0 & 3 \\ 0 & 0 & 4 \\ 0 & & \end{bmatrix}$$

Proof by Gauss-Jordan Elimination

1. Locate the leftmost column that does not consist entirely of zeros
2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \\ 0 & 0 & 5 & \\ 0 & 0 & 2 & \\ 0 & 0 & 9 & \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 5 & 0 & 0 \\ & & \vdots & & \end{bmatrix}$$

Proof by Gauss-Jordan Elimination

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5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix the remains. (Will create REF)

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4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.
5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix the remains. (Will create REF)
6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1s. (Will create RREF)

Exercise

Exercise

Use Gauss-Jordan elimination to put the matrix in RREF

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

and use it to find all homogeneous solutions.

See page 52 of textbook.