SF 1684 Algebra and Geometry
Lecture 3

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Topics for Today

1. Systems of Linear Equations
2. Matrices: Definition and Row Operations

Linear Equations

Definition

An equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where $a_i \in \mathbb{R}$, $b \in \mathbb{R}$ and the x_i are variables is called a **linear equation**.

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Example:

$2x + y = 3$ is a linear equation

$n=2$ $a_1=2$ $x_1=x$ $b=3$

$a_2=1$ $x_2=y$

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If $b = 0$, the equation is then called **homogeneous**. If $b \neq 0$, it is called **non-homogeneous**.

Example:

$$2x + y = 3 \text{ is a linear equation}$$

$$x^2 + 3y = 1 \text{ is not a linear equation}$$

power of two makes it not linear

System of Linear Equations

Definition

Having multiple linear equations

n-variables

m-equation

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

$a_{i,j} \in \mathbb{R}$

$b_j \in \mathbb{R}$

is called an $m \times n$ **system of linear equations**.

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is called an $m \times n$ **system of linear equations**. If all the $b_j = 0$, then the system is called **homogeneous**. If any of $b_j \neq 0$, the system is called **non-homogeneous**.

Determining the solutions (if any) of systems of linear equations is the main motivation behind this whole course.

Example of Problems Using a System of Linear Equations

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Give two lines, L_1 and L_2, is there a point that lies on both lines?

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and L_2 is given by the equation

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and L_2 is given by the equation

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then determining the solutions (if any) to the 2×2 system of linear equations:

2 unknown

2 equation

$$\begin{cases} 2x + 3y = 1 \\ 4x + 6y = 1 \end{cases}$$

would answer our question.

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Given a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, can a new vector $\vec{w}$ be written as a linear combination of the $\vec{v}_i$?

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Example: Let

$$\vec{v}_1 = (1, 2, 3), \vec{v}_2 = (1, 0, 0), \vec{v}_3 = (0, 1, 1), \vec{w} = (1, 5, 3)$$

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The question is now, does there exist an A, B, C such that

$$(1,5,3) = A(1,2,3) + B(1,0,0) + C(0,1,1)$$

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$$(1,5,3) = A(1,2,3) + B(1,0,0) + C(0,1,1) = (A+B, 2A+C, 3A+C)$$

Thus solving the 3×3 system of linear equations

$$\begin{cases} A + B = 1 \\ 2A + C = 5 \\ 3A + C = 3 \end{cases}$$

would answer our question.

Matrix Representation of a Linear System

Given a system of linear equations

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
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the only relevant information are the coefficients $a_{1,1}, a_{1,2}, \ldots$ and the $b_1, b_2, \ldots$.

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the only relevant information are the coefficients $a_{1,1}, a_{1,2}, \ldots$ and the $b_1, b_2, \ldots$. Thus we condense this information into the **matrix of coefficients** and the $\vec{b}$**-vector**

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix} \quad \text{and} \quad \vec{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented Matrix of a Linear System

We also care about how the matrix of coefficients behave with the $\vec{b}$-vector and so we also consider the **augmented matrix**:

$$(A|\vec{b}) := \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} & b_m \end{array}\right)$$

Example

Consider the system of linear equations:

$$\left.\begin{aligned} -2x + 2y + 3z &= 1 \\ 3x + y + 5z &= 7 \\ x + y + z &= 1 \end{aligned}\right\} b$$

Then the matrix of coefficients, $\vec{b}$-vector and augmented matrix, respectively, would be:

$$A = \begin{bmatrix} -2 & 2 & 3 \\ 3 & 1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} \qquad (A|\vec{b}) = \left[\begin{array}{ccc|c} -2 & 2 & 3 & 1 \\ 3 & 1 & 5 & 7 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

Exercise

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Given the augmented matrices

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} \overset{x}{2} & \overset{y}{5} & \overset{z}{5} & 1 \\ 3 & 9 & 6 & 3 \\ 1 & 4 & 5 & 7 \end{array}\right) \qquad (B|\vec{b}) = \left(\begin{array}{ccc|c} \overset{x_1}{1} & \overset{x_2}{0} & \overset{x_3}{0} & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

write down the corresponding system of linear equations.

$$\begin{aligned} 2x + 5y + 5z &= 1 \\ 3x + 9y + 6z &= 3 \\ x + 4y + 5z &= 7 \end{aligned}$$

$$\begin{cases} 1x_1 + 0x_2 + 0x_3 = 3 \Rightarrow x_1 = 3 \\ 0x_1 + 1x_2 + 0x_3 = 4 \Rightarrow x_2 = 4 \\ 0x_1 + 0x_2 + 1x_3 = 5 \Rightarrow x_3 = 5 \end{cases}$$

Terminology

n columns

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

m rows.

A is called an $m \times n$ **matrix**.

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i position

$a_{i,j}$

A is called an $m \times n$ **matrix**.
$m =$ number of rows (equations)
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$a_{i,j}$ is the number in the i^{th} row and j^{th} column.
If $m = n$, we call A a **square matrix**.

3x3

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Terminology 2

Given an augmented matrix

$$(A|\vec{b}) := \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \dots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} & b_m \end{array}\right)$$

we will say a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

solves the augmented matrix if it is a solution to the corresponding system of linear equations.

Example

We say the vector $\vec{x} = (x, y, z) = (-\frac{33}{4}, \frac{9}{4}, \frac{5}{4})$ solves the augmented matrix

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} 2 & 5 & 5 & 1 \\ 3 & 9 & 6 & 3 \\ 1 & 4 & 5 & 7 \end{array}\right)$$

since

$$2x + 5y + 5z = 2\left(-\frac{33}{4}\right) + 5\left(\frac{9}{4}\right) + 5\left(\frac{5}{4}\right) = 1$$

$$3x + 9y + 6z = 3\left(-\frac{33}{4}\right) + 9\left(\frac{9}{4}\right) + 3\left(\frac{5}{4}\right) = 3$$

$$x + 4y + 5z = \left(-\frac{33}{4}\right) + 4\left(\frac{9}{4}\right) + 5\left(\frac{5}{4}\right) = 7$$

Main Motivation

As we stated before one of the main motivations behind this whole course is to find all the solutions (if any) of a given system of linear equations.

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Exercise

Find the solutions to the augmented matrix

$$\left(\begin{array}{ccc|c} -1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 21 \\ 2 & -3 & 2 & 1 \end{array}\right)$$

Exercise Solution

$2x + 2(-x) = 2x - 2x = 0x = 0$

$$\left[\begin{array}{ccc|c} x & y & z & \\ -1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 21 \\ 2 & -3 & 2 & 1 \end{array}\right]$$

$\rightarrow$

$$\begin{array}{ll} E_1 & -x + 2y \quad = 2 \\ E_2 & 2x + y + 2z = 21 \\ E_3 & 2x - 3y + 2z = 1 \end{array} \quad \xrightarrow[E_3 + 2E_1]{E_2 + 2E_1} \quad \begin{array}{l} -x + 2y \quad = 2 \\ 0x + 5y + 2z = 25 \\ 0x + y + 2z = 5 \end{array}$$

$$\begin{array}{l} -x + 2y \quad = 2 \\ 5y + 2z = 25 \\ y + 2z = 5 \end{array} \quad \xrightarrow{E_2 \Leftrightarrow E_3} \quad \begin{array}{l} -x + 2y \quad = 2 \\ y + 2z = 5 \\ 5y + 2z = 25 \end{array} \quad \xrightarrow[E_3 - 5E_2]{E_1 - 2E_2} \quad \begin{array}{l} -x + 0y - 4z = -8 \\ y + 2z = 5 \\ 0y - 8z = 0 \end{array}$$

$$\begin{array}{l} -x \quad -4z = -8 \\ y + 2z = 5 \\ -8z = 0 \end{array} \quad \text{divide } E_3 \text{ by } -8 \quad \begin{array}{l} -x \quad -4z = -8 \\ y + 2z = 5 \\ z = 0 \end{array} \quad \xrightarrow[E_2 - 2E_3]{E_1 + 4E_3} \quad \begin{array}{l} -x \quad = -8 \\ y \quad = 5 \\ z = 0 \end{array}$$

$\rightarrow \quad x = 8, \; y = 5, \; z = 0$

$$\tilde{x} = \begin{bmatrix} 8 \\ 5 \\ 0 \end{bmatrix}$$

Solves the augmented matrix.

Equation Operations

We see that to solve the system of linear equations, we performed certain operations to transform

$$\begin{aligned} -x + 2y &= 2 \\ 2x + y + 2z &= 21 \\ 2x - 3y + 2z &= 1 \end{aligned} \quad \Longrightarrow \cdots \Longrightarrow \quad \begin{aligned} x &= 8 \\ y &= 5 \\ z &= 0 \end{aligned}$$

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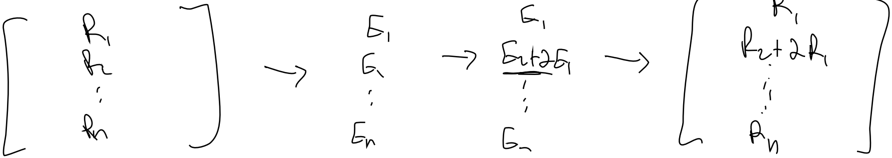
We performed three different types of operations on the equations:

1. Added a multiple of one equation to the other
2. Interchanged two equations
3. Multiplied an equation by a non-zero constant

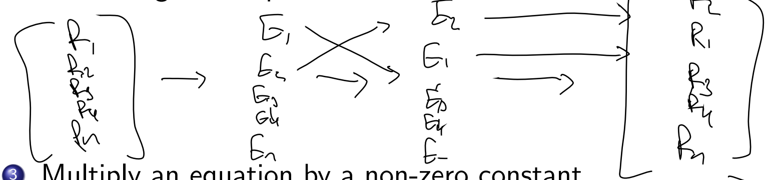
Translate to Matrices

How do these equation operations translate to matrices:

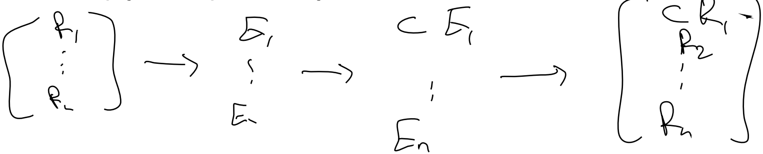
1. Add a multiple of one equation to the other



2. Interchange two equations



3. Multiply an equation by a non-zero constant



Row Operations

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CAUTION!!!!!

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These are *row* operations.

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Definition

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1. Add a multiple of one row to the other
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3. Multiply a row by a non-zero constant

CAUTION!!!!!

These are *row* operations. We can NOT do the same the things to the *columns*!

Row Operations

Definition

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1. Add a multiple of one row to the other
2. Interchange two rows
3. Multiply a row by a non-zero constant

CAUTION!!!!!

These are *row* operations. We can NOT do the same the things to the *columns*!

Can NOT add a multiple of one *column* to the other!

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1. Add a multiple of one row to the other
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3. Multiply a row by a non-zero constant

CAUTION!!!!!

These are *row* operations. We can NOT do the same the things to the *columns*!

Can NOT add a multiple of one *column* to the other!
Can NOT interchange two *columns*!

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1. Add a multiple of one row to the other
2. Interchange two rows
3. Multiply a row by a non-zero constant

CAUTION!!!!!

These are *row* operations. We can NOT do the same the things to the *columns*!

Can NOT add a multiple of one *column* to the other!
Can NOT interchange two *columns*!
Can NOT multiply a *column* by a non-zero constant!

Exercise

Exercise

Use matrices and row operations to find the solution to the system of linear equations

$$x + y + 2z = 9$$

$$2x + 4y - 3z = 1$$

$$3x + 6y - 5z = 0$$

$$\Longrightarrow \begin{bmatrix} x = a \\ y = b \\ z = c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \underset{\text{Row operations}}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

Extra Work Space

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -3.5 & -9.5 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_2]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5.5 & 18.5 \\ 0 & 1 & -3.5 & -9.5 \\ 0 & 0 & 0.5 & 1.5 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 5.5 & 18.5 \\ 0 & 1 & -3.5 & -9.5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 + 3.5R_3]{R_1 - 5.5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow \begin{array}{l} x + 0y + 0z = 2 \\ 0x + 1y + 0z = 1 \\ 0x + 0y + 1z = 3 \end{array}$$

$$\longrightarrow x = 2,\ y = 1,\ z = 3$$

1 1 2
0 2 1 9
0 0 5

Homogeneous Solutions

Definition

Given a matrix A, we say that $\vec{x}$ is a **homogeneous solution** of A if it solves the augmented matrix $(A|\vec{0})$.

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1. *$\vec{0}$ is a homogeneous solution*, $\vec{0}$ is called the trivial (homogeneous) solution
2. *If $\vec{x}$ is a homogeneous solution and $c \in \mathbb{R}$ then $c\vec{x}$ is also a homogeneous solution*
3. *If $\vec{x}$ and $\vec{y}$ are homogeneous solutions than so is $\vec{x} + \vec{y}$*

Proof

1) $\vec{0}$ is a homogenous solution

$$(*): \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ \vdots \qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0 \end{cases}$$

Yes $\vec{0}$ is a solution

2) $\vec{x}$ is a homogenous then $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and solves $(*)$, $c \in \mathbb{R}$

$c\vec{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$, $\quad a_{11}(cx_1) + \dots + a_{1n}(cx_n) = c\,(\underbrace{a_{11}x_1 + \dots + a_{1n}x_n}_{0}) = c \cdot 0 = 0$

$\Rightarrow$ $c\vec{x}$ is a homogeneous solution.

3) $\vec{x}, \vec{y}$ are homogeneous solutions. $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$, $\quad a_{11}(x_1 + y_1) + \dots + a_{1n}(x_n + y_n)$

$= \underbrace{a_{11}x_1 + \dots a_{1n}x_n}_{=0} + \underbrace{a_{11}y_1 + \dots + a_{1n}y_n}_{=0} = 0 + 0 = 0$

$\downarrow$ homogeneous solution.

Non-homogeneous Solution

Theorem

Given an augmented matrix $(A|\vec{b})$ and any vector $\vec{x}_0$ that solves the augmented matrix, then all vectors that solve the matrix will be of the form

$$\vec{x} + \vec{x}_0$$

where $\vec{x}$ is a homogeneous solution of A.

Proof Suppose $(A|\vec{b}) \longrightarrow$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 \cdots a_{mn}x_n &= b_n \end{aligned}$$

$\vec{y}$ is a solution to $(A|\vec{b})$. Want to show that

$\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$\vec{y} = \vec{x} + \vec{x}_0$ where $\vec{x}$ is a homogeneous.

Enough to show $\vec{x} = \vec{y} - \vec{x}_0$ is a homogeneous

Extra Work Space

So if $\vec{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$ & $\vec{X}_0 = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$ is a gen solution.

$$a_{11} Y_1 + \dots + a_{1n} Y_n = b_1 \qquad a_{11} Z_1 + \dots + a_{1n} Z_n = b_1$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1} Y_1 \dots a_{mn} Y_n = b_n \qquad a_{m1} Z_1 + \dots a_{mn} Z_n = b_m$$

$$\vec{Y} - \vec{X}_0 = \begin{bmatrix} Y_1 - Z_1 \\ \vdots \\ Y_n - Z_n \end{bmatrix}$$

$$a_{11}(Y_1 - Z_1) + \dots + a_{1n}(Y_n - Z_n) = \overbrace{a_{11} Y_1 + \dots + a_{1n} Y_n}^{b_1} - \overbrace{(a_{11} Z_1 + \dots + a_{n1} Z_n)}^{b_1}$$

$$= b_1 - b_1 = 0$$

$\Rightarrow \vec{Y} - \vec{X}_0 = \vec{X}$ is a homogeneous $\Rightarrow$ $\vec{Y} = \vec{X} + \vec{X}_0$

0, 1, or ∞ Solution

Theorem

Any augmented matrix $(A|\vec{b})$ *either has*

1. *No solutions*
2. *Exactly* 1 *solution*
3. *Infinitely many solutions*

Extra Work Space