SF 1684 Algebra and Geometry
Lecture 2

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Topics for Today

1. Lines in $\mathbb{R}^n$
2. Planes in $\mathbb{R}^3$
3. Distance between point and line

Lines in $\mathbb{R}^2$

Recall that a line in $\mathbb{R}^2$ can be given by the set of solution to the formula

$$L : Ax + By + C = 0$$

for some values of A, B, C.

Lines in $\mathbb{R}^2$

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$$L : Ax + By + C = 0 \quad \rightarrow \quad By = -Ax - C$$

$$y = \frac{-A}{B}x - \frac{C}{B}$$

for some values of A, B, C.As long as $B \neq 0$, we can rearrange this into the familiar form

$$L : y = mx + b$$

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$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ mx + b \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix} \overset{\text{scalar}}{x} + \begin{bmatrix} 0 \\ b \end{bmatrix} \leftarrow \text{linear combination}$$

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That is, we can write every point as a *linear combination* of the two vectors $\begin{bmatrix} 1 \\ m \end{bmatrix}$ and $\begin{bmatrix} 0 \\ b \end{bmatrix}$.

Parametric Lines in $\mathbb{R}^2$

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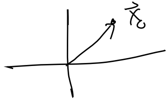

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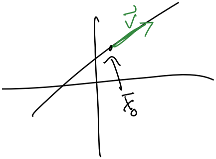

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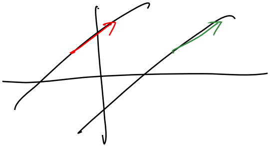
$\vec{v}$ is called a **direction vector**, t is the **parameter** and $\vec{x}_0$ is any point on the line.

Parallel and Perpendicular Lines in $\mathbb{R}^2$

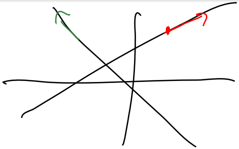
Definition

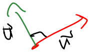
Two lines in $\mathbb{R}^2$ are parallel if and only if their direction vectors are parallel.



Definition

Two lines in $\mathbb{R}^2$ are orthogonal if and only if their direction vectors are orthogonal.





$\longrightarrow \vec{u} \cdot \vec{v} = 0$

Parametric Lines in $\mathbb{R}^n$

Of course, there is nothing stopping us from taking vectors not in $\mathbb{R}^2$. Indeed, for any vectors $\vec{v}, \vec{x}_0 \in \mathbb{R}^n$, we can write the equation of a line in $\mathbb{R}^n$ in parametric form by:

$$\vec{x} = \vec{v}t + \vec{x}_0$$

a point in $\mathbb{R}^n$

a real number

Parametric Lines in $\mathbb{R}^n$

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Example:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{\vec{v}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} t + \underset{\vec{x}_0}{\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}} = \begin{bmatrix} t+2 \\ 2t \\ -t+5 \end{bmatrix}$$

Non-unique Direction Vector

Note that, by it's name, the direction vector $\vec{v}$ only depends on the it's direction. So we may shrink or stretch it as we please and still get the same line.

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Note that, by it's name, the direction vector $\vec{v}$ only depends on the it's direction. So we may shrink or stretch it as we please and still get the same line. Indeed the lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{v} t + \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

$$t=2: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot 2 + \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

and

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \overbrace{\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}}^{v'} t' + \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

$$t'=1 \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

contain the same points as for any value of t, we can just set $t' = t/2$.

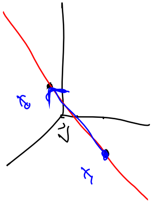
v & v' are parallel so have the same direction.

Exercise

Find the parametric equation for the line going through the points

$$\vec{x}_0 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{x}_1 \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$$

$$\vec{v} = \vec{x}_1 - \vec{x}_0 = \begin{bmatrix} 2 \\ 2 \\ -8 \end{bmatrix}$$



$$\vec{x}_0 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\vec{v} = \vec{x}_0 - \vec{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -2 \\ 8 \end{bmatrix}$$

$$L: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 8 \end{bmatrix} t + \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

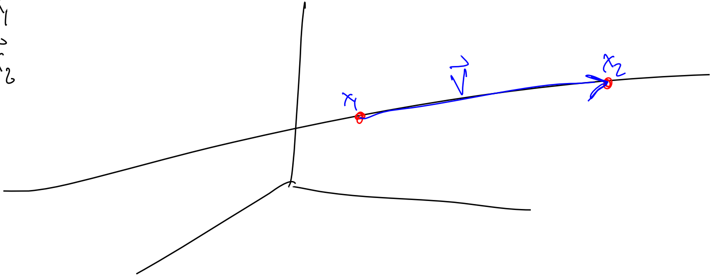
Calculating Direction Vector

In fact, any line passing through the points $\vec{x}_1$ and $\vec{x}_2$ will have a direction vector

$$\vec{v} = \vec{x}_2 - \vec{x}_1$$

$L: \vec{x} = \vec{v}\,t + \vec{x}_1$

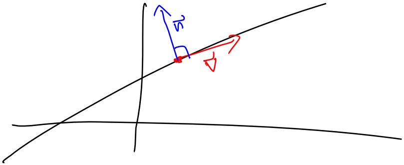
$\vec{x} = \vec{v}\,t + \vec{x}_2$



Normal Vector in $\mathbb{R}^2$

Definition

For any line L with a direction vector $\vec{v}$, we say a vector $\vec{n}$ is **normal** to L if $\vec{n}$ is orthogonal to $\vec{v}$.



Since a direction vector can always be given by $\vec{v} = \vec{x}_2 - \vec{x}_1$ where $\vec{x}_1, \vec{x}_2$ are two point on the line, then we see that

$$\vec{n} \cdot \vec{v} = \vec{n} \cdot (\vec{x}_1 - \vec{x}_2) = 0.$$

In fact, this is an equivalent way to define the equation of the line.

Point-Normal Formula in $\mathbb{R}^2$

Definition

Given a point on a line in $\mathbb{R}^2$, $\vec{x}_0$, and a normal vector $\vec{n}$, then the equation of the line can be given

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If we write $\vec{n} = (n_1, n_2)$ and $\vec{x} = (x, y)$ then we see that $\vec{n} \cdot \vec{x} = n_1 x + n_2 y$.

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$$Ax + By + C = 0$$

number

$$\vec{n} \cdot \vec{x} - \vec{n} \cdot \vec{x}_0 = n_1 x + n_2 y - \vec{n} \cdot \vec{x}_0$$

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point normal formula

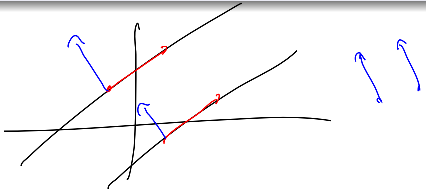
then we can read off a normal for the line as:

$$\vec{n} = \begin{bmatrix} A \\ B \end{bmatrix}$$

Parallel and Perpendicular Lines in $\mathbb{R}^2$ 2

Theorem

Two lines in $\mathbb{R}^2$ are parallel if and only if their normals are parallel.



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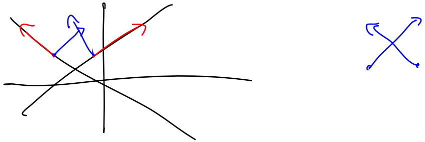

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$$= Ax + By + Cz + \overbrace{(-Ax_0 - By_0 - Cz_0)}^{D}$$

That is, the point normal formula gives us an equation of the form

$$\mathbb{R}^3: \quad Ax + By + Cz + D = 0$$

$$\mathbb{R}^2: \quad Ax + By + C = 0$$

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$$Ax + By + Cz + D = 0$$

What is the geometry of these solutions? Do they form a line?

Plane in $\mathbb{R}^3$

The solutions to the equation

Fix $z = z_0 \Rightarrow Ax + By + Cz_0 + D = 0$

$$Ax + By + Cz + D = 0$$

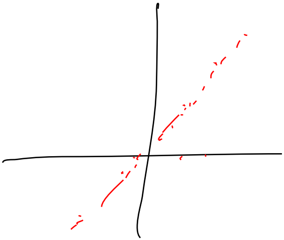
$\Rightarrow Ax + By + C' = 0$

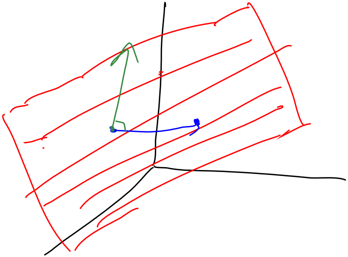
$C' = Cz_0 + D$

form a **plane** in $\mathbb{R}^3$. Moreover, the normal $\vec{n} = (A, B, C)$ is orthogonal to every vector in the plane.

$\mathbb{R}^3: \quad Ax + By + Cz + D = 0$

$\mathbb{R}^2: \quad Ax + By + c = 0$

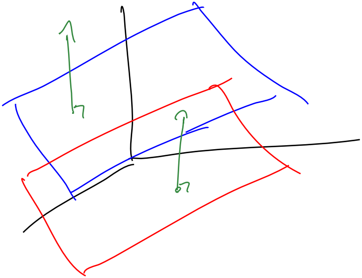


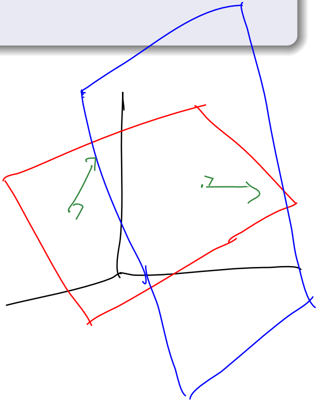


Parallel and Orthogonal Planes in $\mathbb{R}^3$

Definition

Two planes in $\mathbb{R}^3$ are **parallel** if and only if their normals are parallel.
Two planes in $\mathbb{R}^3$ are **orthogonal** if and only if their normals are orthogonal.





Point-Normal Formula in $\mathbb{R}^n$

In general, given an $\vec{n} = (A_1, A_2, \ldots, A_n)$ and an $\vec{x}_0 = (a_1, a_2, \ldots, a_n)$ the point-normal formula gives

$$0 = \vec{n} \cdot \vec{x} - \vec{n} \cdot \vec{x}_0 = A_1 x_1 + A_2 x_2 + \ldots A_n x_n + A_{n+1}$$

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The set of solutions of this equation form what is called an $(n-1)$-dimensional **hyperplane**.

$\mathbb{R}^1 \to$ pt "$\mathbb{R}^0$"

$\mathbb{R}^2 \to$ line "$\mathbb{R}^1$"

$\mathbb{R}^3 \to$ plane "$\mathbb{R}^2$"

$\mathbb{R}^4 \to$ something that "looks like" $\mathbb{R}^3$

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The set of solutions of this equation form what is called an $(n-1)$-dimensional **hyperplane**.

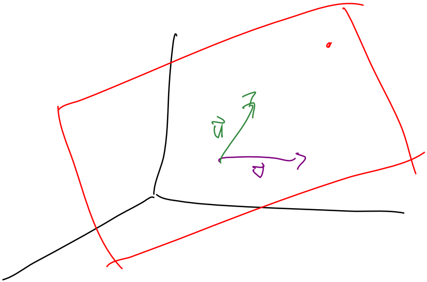
That is, in $\mathbb{R}^n$, the solution set "looks like" $\mathbb{R}^{n-1}$.

Parametric Equation of Plane in $\mathbb{R}^3$

Recall, every point in $\mathbb{R}^2$ can be written as a linear combination of the standard unit vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now, since a plane in $\mathbb{R}^3$ "looks like" $\mathbb{R}^2$ we can write it as a linear combination of two *non-parallel* vectors.

$Pt = s\vec{u} + \vec{v}t$



Definition of Parametric Equation of Plane in $\mathbb{R}^3$

Definition

Given any two non-parallel vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ and a point $\vec{x}_0 \in \mathbb{R}^3$, the **parametric equation** of a plane is

$$\vec{x} = \vec{u} \cdot s + \vec{v} \cdot t + \vec{x}_0, s, t \in \mathbb{R}$$

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Questions:

- Uniques of $\vec{u}, \vec{v}, \vec{x}_0$? No!
- Why must $\vec{u}$,$\vec{v}$ be non-parallel?

$\vec{u} = c\vec{v}$

$\vec{u}s + \vec{v}t = c\vec{v}s + \vec{v}t$

$= \vec{v}(\underbrace{cs + t})$

Example of Parametric Equation of Plane in $\mathbb{R}^3$

Given

$$\vec{u} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} \qquad \vec{x}_0 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

the plane with direction vectors $\vec{u}$, $\vec{v}$ and going through the point $\vec{x}_0$ is given by

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$$= \begin{bmatrix} 2s + 3t \\ 4s - 5t + 2 \\ -2s + 3 \end{bmatrix}$$

Exercise

FACT: Any three point in $\mathbb{R}^3$ that don't all lie on the same line describe a unique plane

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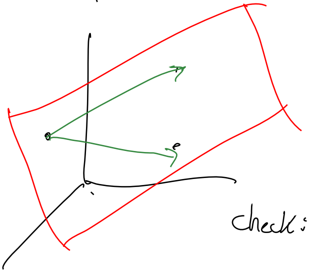
Find the parametric equation of the plane going through the three points $(3, 8, 0)$, $(8, 4, 5)$ and $(5, 0, 9)$

x_0 x_1 x_2



$$\vec{u} = \vec{x}_1 - \vec{x}_0 = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix}$$

$$\vec{v} = \vec{x}_2 - \vec{x}_1 = \begin{pmatrix} 5 \\ 0 \\ 9 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 4 \end{pmatrix}$$

check: $\vec{u}$ & $\vec{v}$ are non-parallel

Plane: $$\vec{x} = \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix} s + \begin{pmatrix} -3 \\ -4 \\ 4 \end{pmatrix} t + \begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix}$$

From Point-Normal Form to Parametric Equation

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4. If $\vec{u}$ and $\vec{v}$ are not parallel, then a parametric equation for your plane will be:

$$\vec{x} = \vec{u} \cdot s + \vec{v} \cdot t + \vec{x}_1$$

From Parametric to Point-Normal Form

Given a plane with parametric equation

$$\vec{x} = \vec{u} \cdot s + \vec{v} \cdot t + \vec{x}_0, \quad s, t \in \mathbb{R}$$

How do you find the point-normal equation?

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First step: find a normal to the plane. That is, find a vector that is orthogonal to every vector on the plane.

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Every vector on the plane will be of the form

$$\vec{u} \cdot s + \vec{v} \cdot t, \quad s, t \in \mathbb{R}$$

So it is enough to find a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$. (Exercise: show both of these statements)

Cross Product

Definition

Given two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Define the **cross product** of $\vec{u}$ and $\vec{v}$ as

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Theorem

For any two vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \times \vec{v}$ is orthogonal to **both** *$\vec{u}$ and $\vec{v}$.*

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CAUTION!!!!!!!

The cross product ONLY works in $\mathbb{R}^3$. This method CANNOT be extended to $\mathbb{R}^n$ for any n except $n = 7$. But even then, the geometry behaves differently.

From Parametric to Point-Normal Form 2

Given a plane in $\mathbb{R}^3$ of the form

$$\vec{x} = \vec{u} \cdot s + \vec{v} \cdot t + \vec{x}_0$$

Calculate a normal of the plane

$$\vec{n} = \vec{u} \times \vec{v}$$

Then the point-normal equation of your plane will be

$$\vec{n} \cdot (\vec{x} - \underline{\vec{x}_0}) = 0$$

Exercise

Find the point-normal equation of the plane going through the three point $(3, 8, 0)$, $(8, 4, 5)$ and $(5, 0, 9)$

$x_0 \qquad x_1 \qquad x_2$

$$\vec{u} = \bar{x}_1 - \bar{x}_0 = \begin{pmatrix}8\\4\\5\end{pmatrix} - \begin{pmatrix}3\\8\\0\end{pmatrix} = \begin{pmatrix}5\\-4\\5\end{pmatrix} \begin{matrix}u_1\\u_2\\u_3\end{matrix} \qquad \vec{v} = \bar{x}_2 - \bar{x}_1 = \begin{pmatrix}5\\0\\9\end{pmatrix} - \begin{pmatrix}8\\4\\5\end{pmatrix} = \begin{pmatrix}-3\\-4\\4\end{pmatrix} \begin{matrix}v_1\\v_2\\v_3\end{matrix}$$

Check: u & v are not parallel!!

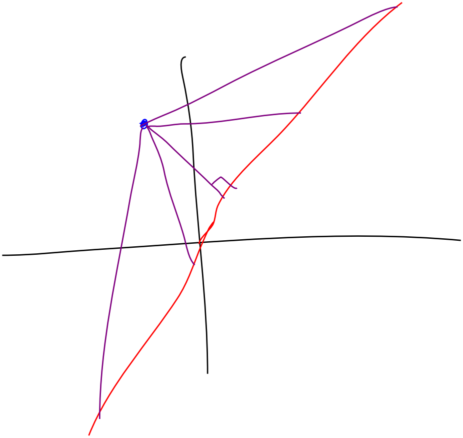
$$\vec{n} = \vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \begin{bmatrix} -4 \cdot 4 - 5 \cdot -4 \\ -(5 \cdot 4 - 5 \cdot -3) \\ 5 \cdot -4 - (-4)(-3) \end{bmatrix} = \begin{bmatrix} 4 \\ -35 \\ -32 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

$$\vec{n} \cdot \vec{x} - \vec{n} \cdot \vec{x}_0 = 4x - 35y - 32z - (4 \cdot 3 - 35 \cdot 8 - 32 \cdot 0) = 0$$

$$4x - 35y - 32z + 268 = 0$$

Shortest Distance Between a Point and a Line in $\mathbb{R}^2$

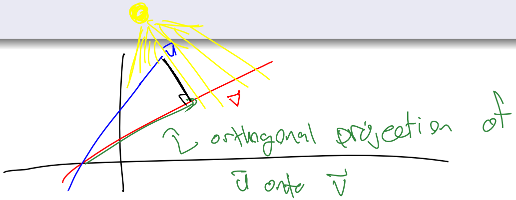
Given a point (x_0, y_0) and a line that goes through the origin $L : \vec{x} = \vec{v}t$, what is the shortest distance between the point and the line?



Orthogonal Projection

Definition (Informal)

The **orthogonal projection** of a vector $\vec{u}$ onto a vector $\vec{v}$ is the “shadow” of $\vec{u}$ on $\vec{v}$.



Definition (Formal)

The **orthogonal projection** of a vector $\vec{u}$ onto a vector $\vec{v}$ is

$$\underline{\mathrm{proj}_{\vec{v}}\vec{u}} = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\vec{v}$$

$\vec{u}\cdot\vec{v}$ is a number

$||\vec{v}||^2$ is also a number!

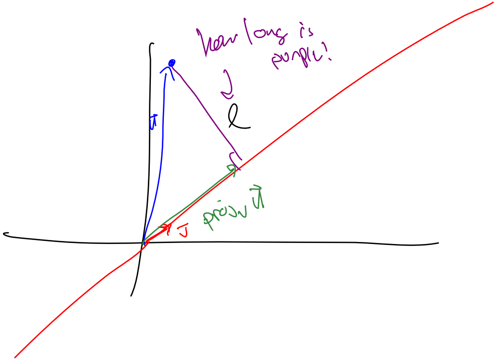
Exercise: Show that these definitions are same. Hint: $\mathrm{proj}_{\vec{v}}\vec{u}$ must be parallel to $\vec{v}$ but $\vec{u} - \mathrm{proj}_{\vec{v}}\vec{u}$ must be orthogonal to $\vec{v}$.

Shortest Distance Between a Point and a Line in $\mathbb{R}^2$ 2

Theorem

The shortest distance between a point $\vec{u} = (u_1, u_2)$ and the line passing through the origin $L : \vec{x} = \vec{v}t$ will be

$$\sqrt{||\vec{u}||^2 - ||proj_{\vec{v}}\vec{u}||^2}$$



$$||\vec{u}||^2 = ||proj_{\vec{v}}\vec{u}||^2 + \ell^2$$

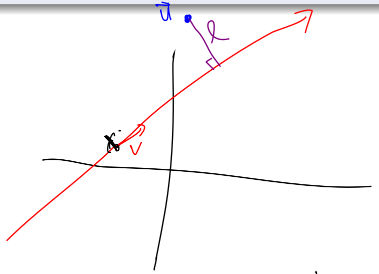
$$\ell^2 = ||u||^2 - ||proj_{\vec{v}}\vec{u}||^2$$

$$\ell = \sqrt{||u||^2 - ||proj_{\vec{v}}\vec{u}||^2}$$

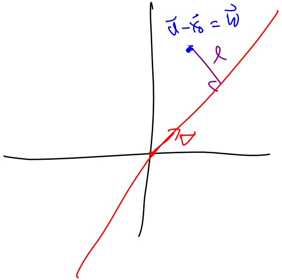
Theorem

The shortest distance between a point $\vec{u} = (u_1, u_2)$ and the line $L : \vec{x} = \vec{v}t + \vec{x}_0$ will be the same as the shortest distance between the point $\vec{w} = \vec{u} - \vec{x}_0 = (u_1 - x_0, u_2 - y_0)$ and the line passing through the origin $L' : \vec{x} = \vec{v}t$:

$$\sqrt{||\vec{w}||^2 - ||proj_{\vec{v}}\vec{w}||^2}$$



subtract $\vec{x}_0$



apply previous thm
to $\vec{w}$ & L'