SF 1684 Algebra and Geometry
Lecture 1

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Course Outline

Structure of the course: FFÖFÖS

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Seminar problems:

- Posted on Mondays
- Hand in answers following Monday during the seminar
- Get solutions from TA there (no physical solutions will be given)

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Bonus points:

- 1 random question on each seminar will be graded
- The clarity and readability of your solution will also be graded
- 1 bonus point will be awarded for correct seminar assignment (total of 6)
- Bonus points can be used *only* for the first question on the exam

Topics for Today

- Vectors
- Vector Spaces: Axioms, $\mathbb{R}^n$
- Relations on $\mathbb{R}^n$: Norm, dot product, orthogonality

Vectors

Definition

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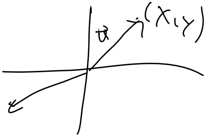
5km/h North

An example of a vector would be velocity: a *speed* with a *direction*.

Another example would be an arrow on the cartesian plane. These can be represented by the end point of the arrow (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$.

take values in our field



Scalars

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In this course only consider reals.

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Some examples: $\mathbb{Q}$ (rationals), $\mathbb{R}$ (reals) or $\mathbb{C}$ (complex numbers).

Definition

The elements of the field over which our vector space is defined are called **scalars**.

In this course, the scalars are always real numbers.

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addition exists and makes sense.

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8. (Associativity) For ever $c, d \in F$ and every $\vec{u} \in V$, $c \cdot (d \cdot \vec{u}) = (cd) \cdot \vec{u}$

* (bracketing items 6–8)

* we can multiply by our base field ($\mathbb{R}$) in a way that makes sense.

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* vector addition and scalar multiplication makes sense.

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9. * (Distributivity) For every $c, d \in F$ and every $\vec{u}, \vec{v} \in V$, $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{v}$ and $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$

Vector in $\mathbb{R}^n$

We denote the set of vectors

$$\mathbb{R}^n = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, x_i \in \mathbb{R} \right\}.$$

→ set of ordered tuples of length n in $\mathbb{R}$.

If $n=2$: points on the cartesian plane.

If $n=3$: points in 3d-space.

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$\mathbb{R}^n$ is a Vector Space

Theorem

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Exercise

Check that all the axioms are satisfied when we set

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad -\vec{x} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$$

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Note that even though we did not define it as such we get that

$$-\vec{x} = (-1) \cdot \vec{x}$$

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Show that $\{f : \mathbb{R} \to \mathbb{R}\}$ is a vector space. What is the 0-vector? What is a vectors negative? What is a scalar? What is scalar multiplication?

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Unless otherwise stated, the vector space we work with will be $\mathbb{R}^n$ for some n.

Examples

Vectors in $\mathbb{R}^2$ are arrows:

$$\vec{u} = (-1, 2) \quad \vec{v} = (3, 4)$$

Addition: placing one arrow at the tip of the other

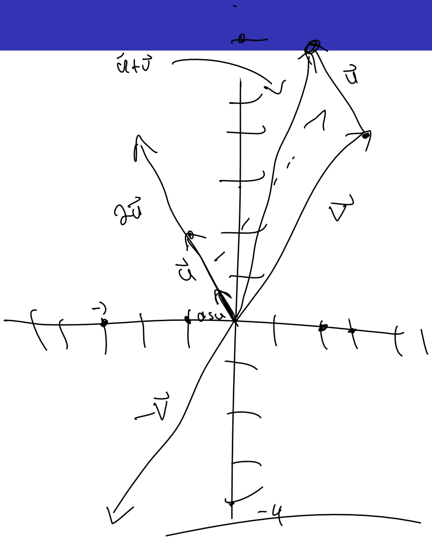
$$\vec{u} + \vec{v} = (2, 6)$$

Negation: changing the direction of the arrow

$$-\vec{v} = (-3, -4)$$

Scalar multiplication: strecthing or shrinking the arrow:

$$2\vec{u} = (-2, 4) \quad 0.5\vec{u} = (-0.5, 1)$$



Parallel and Norm

Definition

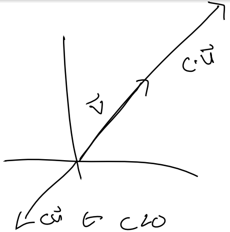
We say that $\vec{u}$ and $\vec{v}$ are **parallel** if there is a scalar c such that $\vec{u} = c \cdot \vec{v}$.

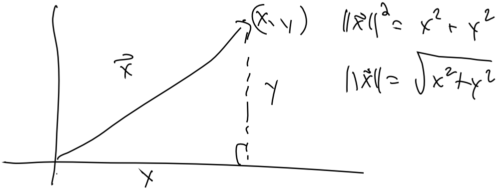
Definition

For $\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ we define the **norm** of $\vec{x}$ as

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}.$$

We can think of the norm of $\vec{x}$ as the "length of the arrow".





Properties of the Norm

Exercise

If $\vec{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

1. $\|\vec{x}\| \geq 0$
2. $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$
3. $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$

1) $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \geq 0$ because $\sqrt{\ }$ is always positive (if it exists)

The square root always makes sense as $x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$

2) $\|\vec{x}\| = 0 \iff x_1^2 + \dots + x_n^2 = 0 \iff x_1^2 = 0, x_2^2 = 0, \dots x_n^2 = 0 \iff x_1 = 0, \dots x_n = 0$
$\iff \vec{x} = \vec{0}$

3) $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $c\vec{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$ $\|c\vec{x}\| = \sqrt{(cx_1)^2 + \dots + (cx_n)^2} = \sqrt{c^2(x_1^2 + \dots + x_n^2)}$
$= \sqrt{c^2}\,\|\vec{x}\| = |c|\,\|\vec{x}\|$

Distance Between Two Vectors

The distance between two vectors is the "distance between the tips of the arrows".

Thus we see that the distance between the two vectors $\vec{u}$ and $\vec{v}$ will be the length of $\vec{v} - \vec{u}$.

Hence, we may define

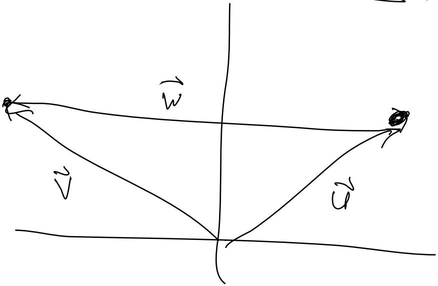
$$d(\vec{u}, \vec{v}) := \|\vec{v} - \vec{u}\|$$

Exercise: Show that $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$.

$\vec{v} = \vec{u} + \vec{w}$

$\rightarrow \vec{w} = \vec{v} - \vec{u}$

$d(u, v) = \|\vec{w}\| = \|\vec{v} - \vec{u}\|$

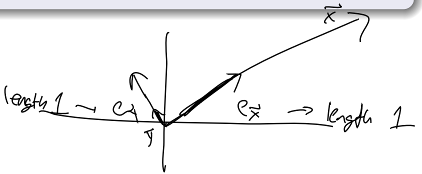


Definition

We say a vector $\vec{x}$ is a **unit vector** if $\|\vec{x}\| = 1$

For any vector $\vec{x}$, the vector

$$\vec{e}_{\vec{x}} = \frac{1}{\|\vec{x}\|}\vec{x}$$



is a unit vector. Moreover $\vec{e}_{\vec{x}}$ is parallel to $\vec{x}$.
By moving from $\vec{x}$ to $\vec{e}_{\vec{x}}$ we say we have **normalized** $\vec{x}$ and say that $\vec{e}_{\vec{x}}$ is the **normalization** of $\vec{x}$.
We denote

$$\vec{e}_1 = (1, 0, 0, \ldots, 0), \vec{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \vec{e}_n = (0, 0, \ldots, 0, 1).$$

Then each $\vec{e}_i$ is a unit vector and we call them the **standard unit vectors**.

Exercises

Let $\vec{u} = (4, 4, -2)$, $\vec{v} = (2, 2, 1)$

1. Are $\vec{u}$ and $\vec{v}$ parallel?
2. Find the distance between $\vec{u}$ and $\vec{v}$.
3. Find a unit vector that is parallel to $\vec{v}$.

1) $\vec{u}$ & $\vec{v}$ parallel if $\vec{u} = c\vec{v}$

$(4,4,-2) = \vec{u} = c\vec{v} = (2c, 2c, 1)$

Need $2c = 4$, $2c = 4$, $c = -2$

No such c!

2) $d(\vec{u},\vec{v}) = \|\vec{v} - \vec{u}\| = \|(-2, -2, 3)\|$

$= \sqrt{(-2)^2 + (-2)^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$

3) $\|v\| = \sqrt{2^2+2^2+1} = 3$

$\vec{e}_{\vec{v}} = \frac{1}{3}\vec{v} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ is a unit vector which is parallel to $\vec{v}$

Linear Combinations

We say $\vec{u}$ is a linear combination of the m vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ if there exists m scalars $a_1, a_2, \ldots, a_m$ such that

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m.$$

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$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m.$$

Every vector $\vec{x} \in \mathbb{R}^n$ can be written as a linear combination of the standard unit vectors $\vec{e}_1, \ldots, \vec{e}_n$. Indeed, if $\vec{x} = (x_1, \ldots, x_n)$, then we see

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

where we recall that

i-th position

$$\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

Dot Product in $\mathbb{R}^n$

For

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

we define the **dot product** of the two vectors as

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

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Observe that if we let $\vec{x} = \vec{y}$, then we get that

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Properties of the Dot Product

Exercise:

prove these.

If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

1. $\vec{x} \cdot \vec{x} \geq 0$
2. $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$.
3. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
4. $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$ ⟶ is essentially the same as distributing multiplication into addition in regular algebra.
5. $(c\vec{x}) \cdot \vec{y} = \vec{x} \cdot (c\vec{y}) = c(\vec{x} \cdot \vec{y})$

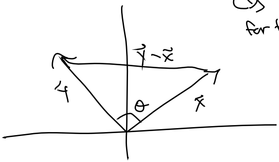
Theorem of the Dot Product

Theorem (Section 1.2)

For $\vec{x}, \vec{y} \in \mathbb{R}^n$

1. $\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos(\theta)$ *where* θ *is the angle between the two vectors.*
2. $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$ *(Cauchy-Schwartz inequality)*
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ *(Triangle Inequality)*

1)



cos-law for triangles $\|\vec{y}-\vec{x}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

$\|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$, $\|\vec{y}\|^2 = \vec{y}\cdot\vec{y}$

$\|\vec{y}-\vec{x}\|^2 = (\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) = \vec{y}\cdot\vec{y} - \vec{y}\cdot\vec{x} - \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{x}$

$= \vec{y}\cdot\vec{y} - 2\vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{x}$

$\vec{y}\cdot\vec{y} - 2\vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{x} = \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} - 2\|\vec{x}\|\|\vec{y}\|\cos\theta \rightarrow -2\vec{x}\cdot\vec{y} = -2\|\vec{x}\|\|\vec{y}\|\cos\theta$

$\vec{x}\cdot\vec{y} = \|\vec{x}\|\cdot\|\vec{y}\|\cos\theta$

Proofs

2) Show $|\vec{x}\cdot\vec{y}| \le \|\vec{x}\|\cdot\|\vec{y}\|$

Proof: $|\vec{x}\cdot\vec{y}| = \big|\|x\|\cdot\|\vec{y}\|\cos\theta\big| = \|\vec{x}\|\,\|\vec{y}\|\,|\cos\theta| \le \|\vec{x}\|\cdot\|\vec{y}\|$

3) Show $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$

Proof: $\|\vec{x}+\vec{y}\| = \sqrt{(\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})} = \sqrt{\vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}}$

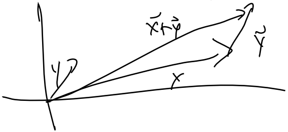
(*) $\le \sqrt{|\vec{x}\cdot\vec{x}| + 2|x\cdot y| + |\vec{y}\cdot\vec{y}|}$

(part 2) $\le \sqrt{\|x\|\cdot\|x\| + 2\|x\|\,\|y\| + \|y\|\cdot\|y\|}$

$= \sqrt{\|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2}$

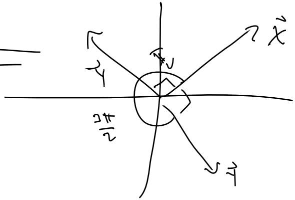
$= \sqrt{(\|x\| + \|y\|)^2}$ $\rightarrow$ $= \|\vec{x}\| + \|\vec{y}\|$

for all $a \in \mathbb{R}$, $a \le |a|$ (*)



Orthogonal Vectors

$0 = \vec{x} \cdot \vec{y}$



From the first part of the theorem, we see that

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos(\theta)$$

and since $\|\vec{x}\|$ and $\|\vec{y}\|$ is never 0 (unless $\vec{x}$ or $\vec{y}$ themselves were $\vec{0}$), we get that

0, the number

$\vec{0}$, the vector

$$\vec{x} \cdot \vec{y} = 0 \text{ if and only if } cos(\theta) = 0 \text{ if and only if } \theta = \pi/2 \text{ or } 3\pi/2.$$

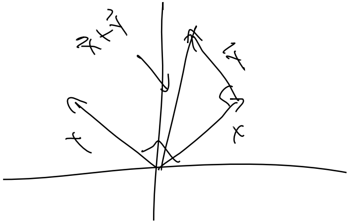
Definition

Two vectors $\vec{x}$ and $\vec{y}$ are said to be **orthogonal** if $\vec{x} \cdot \vec{y} = 0$.

Pythagorean Theorem

Theorem

If $\vec{x} \cdot \vec{y} = 0$ *then* $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$.



$$\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y})$$

$$= \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y}$$

$$= \|x\|^2 + 0 + 0 + \|y\|^2$$

$$= \|x\|^2 + \|y\|^2$$

Exercise

1. Let

exercise

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Find the angle between them.

2. Find a vector orthogonal to

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

.

2) if $\begin{bmatrix} x \\ y \end{bmatrix}$ orth to $\vec{u}$

the $\vec{u} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$x + 2y = 0$

let $x = 2$ & $y = -1$

the $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ orth to $\vec{u}$.