QR-method lecture 1

SF2524 - Matrix Computations for Large-scale Systems

So far we have in the course learned about...

Methods suitable for large sparse matrices

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  - Outer isolated eigenvalues
  - Only requires matrix vector products Ay
  - Underlying the matlab command: `eigs`

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- Underlying the matlab command: `eig`
- Computes all eigenvalues
- Suitable for dense problems
- Small matrices in comparison to previous algorithms

Agenda QR-method

1. Decompositions
    - Jordan form
    - Schur decomposition
    - QR-factorization
2. Basic QR-method
3. Improvement 1: Two-phase approach
    - Hessenberg reduction
    - Hessenberg QR-method
4. Improvement 2: Acceleration with shifts
5. Convergence theory

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Reading instructions

Point 1: TB Lecture 24 (background.pdf)
Points 2-4: Lecture notes (qrmethod.pdf)
Point 5: Lecture notes (qrmethod.pdf) (TB Chapter 28)
(Extra reading: TB Chapter 25-26, 28-29)

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Similarity transformation

Suppose $A \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{n\times n}$ is an invertible matrix. Then

$$A$$

and

$$B = VAV^{-1}$$

have the same eigenvalues.

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Numerical methods based on similarity transformations

- If B is triangular we can read-off the eigenvalues from the diagonal.
- Idea of numerical method: Compute V such that B is triangular.

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where

$$\Lambda = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix},$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}, \quad i = 1, \ldots, k$$

Common case: distinct eigenvalues

Suppose $\lambda_i \neq \lambda_j$, $i = 1, \ldots, n$. Then,

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Common case: symmetric matrix

Suppose $A = A^T \in \mathbb{R}^{n \times n}$. Then,

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Example - numerical stability of Jordan form

Consider

$$A = \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ \varepsilon & & 2 \end{pmatrix}$$

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If $\varepsilon > 0$. Then, the eigenvalues are distinct and

$$\Lambda = \begin{pmatrix} 2 + O(\varepsilon^{1/3}) & & \\ & 2 + O(\varepsilon^{1/3}) & \\ & & 2 + O(\varepsilon^{1/3}) \end{pmatrix}.$$

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$\Rightarrow$ JCF not continuous with respect to ε

$\Rightarrow$ JCF is often not numerically stable

Schur decomposition (essentially TB Theorem 24.9)

Suppose $A \in \mathbb{C}^{n\times n}$. There exists an unitary matrix P

$$P^{-1} = P^*$$

and a triangular matrix T such that

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The Schur decomposition is numerically stable.
Goal with QR-method: Numercally compute a Schur factorization

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QR-factorization

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Note: Very different from Schur factorization

$$A = PTP^*$$

- QR-factorization can be computed with a finite number of operations
- Schur decomposition directly gives us the eigenvalues

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Didactic simplifying assumption: All eigenvalues are real

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Simple basic idea: Let $A_0 = A$ and iterate:

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- Set $A_{k+1} = RQ$.

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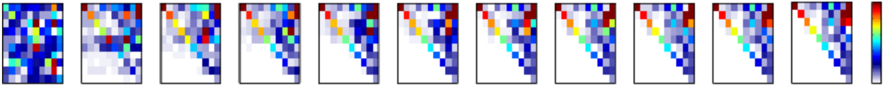
- $A_1 = RQ = Q^* A_0 Q \Rightarrow A_0, A_1, \ldots$ have the same eigenvalues
- More remarkable: $A_k \to$ triangular matrix (except special cases)

$A_k \to$ triangular matrix:

$A_k \rightarrow$ triangular matrix:



* Time for matlab demo *

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Disadvantages

- Computing a QR-factorization is quite expensive. One iteration of the basic QR-method

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- The method often requires many iterations.

Improvement demo:

`http://www.youtube.com/watch?v=qmgxzsWWsNc`