Lecture 15 part II.

Another relative duality:

Theorem. Let $K \subset M$ be a compact, locally contractible subset of a closed (compact, no boundary), oriented n-manifold.

Then $H^k(K) \cong H_{n-k}(M, M-K)$

pf: Let $\mathcal{U}$ be a neighborhood of K in M.

M

Then $M = U \cup (M-K)$

Mayer-Vietoris:

$$\cdots \to H_n(\mathcal{U}) \oplus H_n(M-K) \to \underset{[M]}{H_n(M)} \to H_{n-1}(\mathcal{U}-K) \to$$

$$0 \to \underset{[\mathcal{U}]}{H_n(\mathcal{U},\mathcal{U}-K)} \oplus \underset{[M-K]}{H_n(M-K,\mathcal{U}-K)} \xrightarrow{\cong} H_n(M,\mathcal{U}-K) \to 0$$

Then we have a long exact ladder:

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(M-K) \arrow[r] & H_i(M) \arrow[r] & H_i(M,M-K) \cong H_i(\mathcal{U},\mathcal{U}-K) \arrow[r] & H_{i-1}(M-K) \to \cdots \\
& H^{n-i}(M-K,\mathcal{U}-K) \arrow[u, "{[M-K]\frown}"] & & & \\
\cdots \arrow[r] & H^{n-i}(M,\mathcal{U}) \arrow[u, "\cong \text{ exc.}"] \arrow[r] & H^{n-i}(M) \arrow[uu, "{[M]\frown}", "\cong"'] \arrow[r] & H^{n-i}(\mathcal{U}) \arrow[uu, "{[\mathcal{U}]\frown}"] \arrow[r] & H^{n-i+1}(M,\mathcal{U}) \arrow[uu] \to
\end{tikzcd}$$

Now take $\operatorname{colim}_{\mathcal{U} \supseteq K}$, $\mathcal{U}$ smaller and smaller neighborhoods of K.

* becomes

$$\begin{matrix} H_i(M-K) \\ \uparrow \cong \\ H_c^{n-i}(M-K) \end{matrix} \quad \text{(noncompact Poincaré duality)}$$

$\Rightarrow$ 5-lemma $\Rightarrow$ ※ $\operatorname{colim}_{\mathcal{U} \supseteq K} H^{n-i}(\mathcal{U}) \cong H_i(M, M-K)$

$\downarrow$ isomorphic?

"Čech cohomology" $H^{n-i}(K)$ (reading homework)

Corollary (Alexander duality) ∎

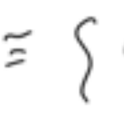
Let $\emptyset \neq K \subsetneq S^n$ be compact, locally contractible.

Then $\tilde{H}^k(K) \cong \tilde{H}_{n-k-1}(S^n - K)$

pf (sketch) $H_i(S^n, S^n - K) \cong \tilde{H}_{i-1}(S^n - K)$ except in degree $i = n$, taken care of by taking reduced cohomology $\tilde{H}(K)$

∎

Examples. Let $S^1 \hookrightarrow S^3$ be a knot.



$$\tilde{H}^k(S^1) = \begin{cases} \mathbb{Z} & ; k=1 \\ 0 & ; \text{o/w} \end{cases}$$

$$\Rightarrow \tilde{H}_k(S^3 - S^1) \cong \begin{cases} \mathbb{Z} & ; k=1 \\ 0 & ; \text{o/w.} \end{cases}$$

independent of the embedding.

On the other hand,

Thm (1989) Gordon-Luecke: if two knots are non-isotopic then their complements are not homotopy equivalent

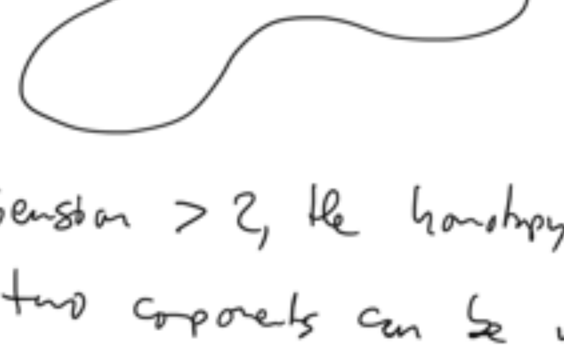
So $S^3 - S^1$ might have a complicated π_1, but $(\pi_1)^{ab} = H_1 \cong \mathbb{Z}$.

Ex: Let $f: S^{n-1} \hookrightarrow S^n$ be any embedding.

Then
$$\begin{matrix} \tilde{H}^{n-1}(S^{n-1}) & \underset{\text{Alex}}{\cong} & \tilde{H}_0(S^n - S^{n-1}) \\ \wr\wr & & \wr\wr \\ \mathbb{Z} & & \mathbb{Z} \end{matrix}$$

So $S^n - S^{n-1}$ has two components

$\Rightarrow$ generalized Jordan curve theorem (Jordan-Schoenfliess thm)



(In dimension > 2, the homotopy type of these two components can be wild!)

"Alexander horned sphere"

Corollary. $X \subseteq \mathbb{R}^n$, X compact, locally contractible.

Then $H_k(X) = \begin{cases} 0, & k \geq n \\ \text{torsion free}, & k = n-1 \text{ or } n-2. \end{cases}$

Pf: $\tilde{H}_k(S^n - X) \cong \tilde{H}^{n-1-k}(X)$

$\tilde{H}_k(S^n - X) = \begin{cases} 0 ; & k < 0 \\ \text{torsion free} & k = 0 \end{cases}$

so $\tilde{H}^i(X) = \begin{cases} 0 ; & i \geq n \\ \text{torsion free} & i = n-1 \end{cases}$

Look at the universal coefficient sequence:

$$0 \to \operatorname{Ext}(H_{k-1}(X); \mathbb{Z}) \to H^k(X) \to \overbrace{\operatorname{Hom}(H_k(X); \mathbb{Z})}^{\text{always torsion free}} \to 0$$

$k \geq n$ $\qquad 0$

$k = n-1$ $\qquad$ torsion free

so $H_k(X) = 0 \qquad k \geq n$

$\operatorname{Ext}(H_{k-1}(X); \mathbb{Z}) = 0 \Rightarrow H_{k-1}(X)$ is torsion free

$\operatorname{Ext}(H_{k-2}(X); \mathbb{Z})$ torsion free $\Rightarrow H_{k-2}(X)$ is torsion free.

Cor: Closed, nonorientable manifolds cannot be embedded in codimension 1. (since $H_{n-1}(M) \supseteq \mathbb{Z}/2\mathbb{Z}$)

e.g. $\mathbb{RP}^n \not\hookrightarrow \mathbb{R}^{n+1}$