# Algebraic Topology

## Lecture 15

**Recall.** $n$-dim. manifold with boundary

$=$ $2^{nd}$ countable Hausdorff space locally homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$.

e.g.

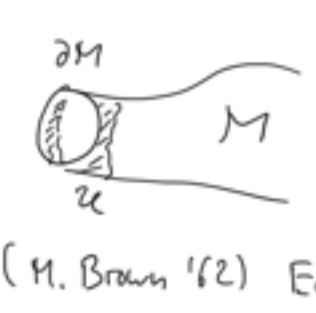
Boundary $\partial M = \bigcup_{\substack{\varphi: U \to \mathbb{R}^n_+ \\ \text{charts}}} \varphi^{-1}(\mathbb{R}^{n-1} \times \{0\})$

$= \{x \in M \mid H_n(M, M - \{x\}) = 0\}$

$\partial M$ is an $(n-1)$-dim manifold without boundary.

**Def.** A collar of a manifold with boundary is an open neighborhood $U$ of $\partial M$ in $M$ and a homeomorphism

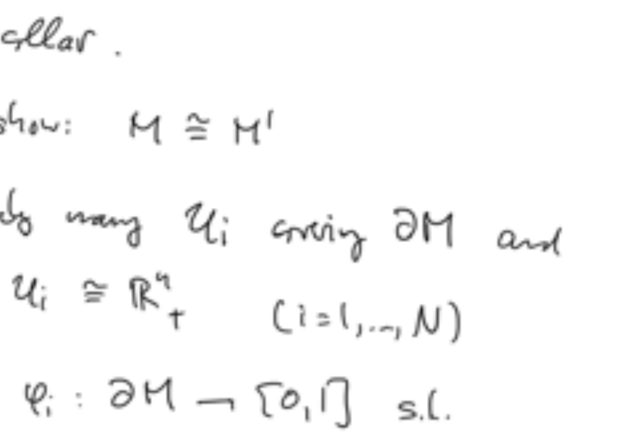
$$\begin{array}{ccc} U & \xrightarrow{\cong} & \partial M \times [0,1) \\ \cup & \circlearrowleft & \cup \\ \partial M & = & \partial M \times \{0\} \end{array}$$




**Thm** (M. Brown '62) Every manifold has a collar.

**Pf.** will show proof for compact $M$.

Let $M' = M \cup_{\partial M} \partial M \times [0,1]$



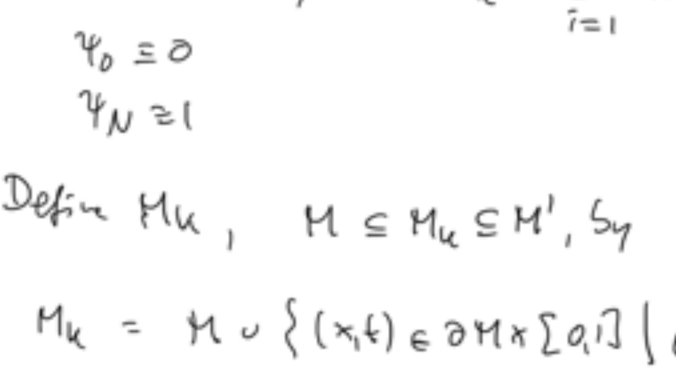

It is clear that $M'$ has a collar.

Enough to show: $M \cong M'$

Choose finitely many $U_i$ covering $\partial M$ and such that $U_i \cong \mathbb{R}^n_+$ $(i = 1, \dots, N)$

Choose maps $\varphi_i : \partial M \to [0,1]$ s.t.

- $\operatorname{supp}(\varphi_i) \subseteq U_i$ $\quad \left( \operatorname{supp}(\varphi_i) = \overline{\varphi_i^{-1}((0,1])} \right)$
- $\operatorname{supp}(\varphi_i)$ cover $\partial M$
- $\sum_{i=1}^{N} \varphi_i \equiv 1$ $\quad$ ("partition of unity")




For $0 \leq k \leq N$, let $\psi_k = \sum_{i=1}^{k} \varphi_i$

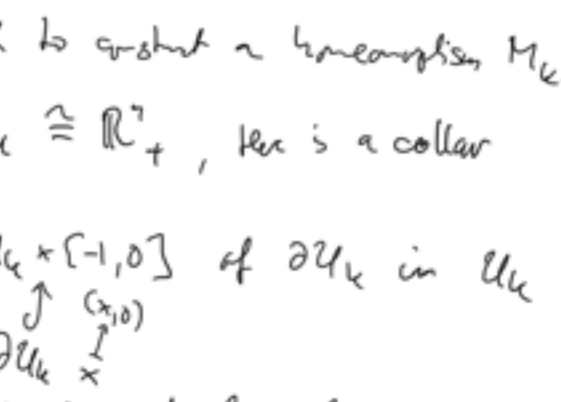
$\psi_0 \equiv 0$

$\psi_N \equiv 1$

Define $M_k$, $M \subseteq M_k \subseteq M'$, by

$$M_k = M \cup \{(x,t) \in \partial M \times [0,1] \mid t \leq \psi_k(x)\}$$

$$M = M_0 \subset M_1 \subset M_2 \subset \dots \; M_N = M'$$



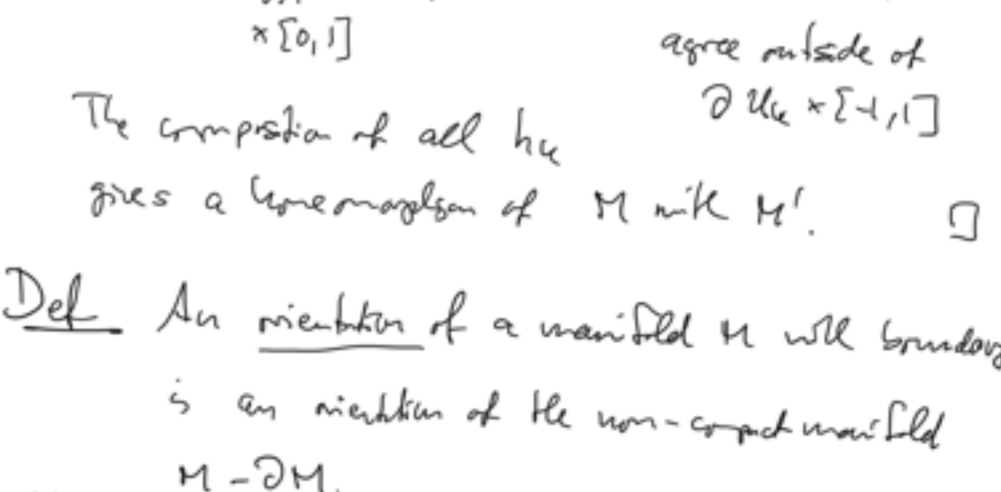

It's enough to construct a homeomorphism $M_{k-1} \xrightarrow{h_k} M_k$.

Since $U_k \cong \mathbb{R}^n_+$, there is a collar

$$\begin{array}{ccc} \partial U_k \times [-1,0] & \text{of } \partial U_k \text{ in } U_k \\ \cup & (x,0) \\ \partial U_k & x \end{array}$$

Gluing with the external collar gives $\partial U_k \times [-1,1] \subseteq M'$

Define $h_k$ like this:




$M_{k-1}$ and $M_k$ agree outside of $\partial U_k \times [-1,1]$

The composition of all $h_k$ gives a homeomorphism of $M$ with $M'$. $\square$

**Def.** An orientation of a manifold $M$ with boundary is an orientation of the non-compact manifold $M - \partial M$.

$\frac{M \text{ compact, orientable}}{\text{Collar Thm}} \Rightarrow$ $\quad H_n(M, \partial M; R) \cong H_n(M - \partial M, \partial M \times (0,1))$

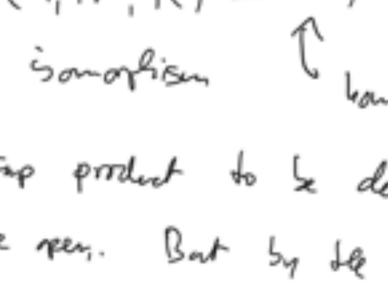
$\qquad [M] \in H_n(M, \partial M; R)$

$\qquad \cong H_n(\underbrace{M - \partial M}_{\text{noncompact manifold}}, \underbrace{\partial M \times \{\tfrac{1}{2}\}}_{\text{compact subset}})$ $\ni$ orientation class

**Thm** (Poincaré duality Thm for manifolds with boundary)

Let $(M, \partial M)$ be a compact manifold

$\partial M = A \cup B$ $\quad$ $A, B$ $n$-dim. manifolds with boundary s.t. $\partial A = \partial B = A \cap B$.




Then

$$D_M : H^k(M, A; R) \xrightarrow{[M] \cap -} H_{n-k}(M, B; R)$$

is an isomorphism $\quad$ however B. case 1.

**Pf.:** For the cap product to be defined, $A$ & $B$ should be open. But by the collar thm, they are homotopy equivalent to open subsets, and that is enough.

**Case $B = \emptyset$.**

$$\begin{array}{ccc} H^k(M, \partial M; R) & \xrightarrow{[M] \cap -} & H_{n-k}(M; R) \\ \cong & \circlearrowleft & \cong \\ H^k_c(M - \partial M; R) & \xrightarrow[\text{noncompact}]{} & H_{n-k}(M - \partial M; R) \end{array}$$

collar thm

Poincaré duality theorem

Explanation: $\quad H^k_c(M - \partial M) = \operatorname{colim}_{K \text{ compact}} H^k(M - \partial M, M - \partial M - K)$

**Case $B \neq \emptyset$:**

l.e.s of triple $(M, \partial M, A)$: $\quad H^k(M, \partial M) \to H^k(M, A) \to H^k(\partial M, A) \to H^{k+1}(M, \partial M)$

$B = \emptyset$ case $\cong$ $\quad \Rightarrow \cong$ $[M] \cap -$ $\quad$ $\cong \downarrow$ exc. $\quad H^k(B, \partial B)$ $\quad \downarrow \cong$ $\quad$ $D = \emptyset$ case $\quad \cong$

l.e.s. of pair $(M, B)$: $\quad H_{n-k}(M) \to H_{n-k}(M, B) \to H_{n-k-1}(B) \to H_{n-k-1}(M)$

one dimension

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