Lecture 14, part II

Sketch proof of Lemma.

Take $K \subset U$, $L \subset V$ compact subsets.

$$\begin{tikzcd}[column sep=small]
\to H^k(M, M-(K\cap L)) \arrow[r] \arrow[d, "\cong"] & H^k(M,M-K)\oplus H^k(M,M-L) \arrow[r] \arrow[d, "\cong"] & H^k(M,M-(K\cup L)) \arrow[r, "\delta^*"] \arrow[d, "\cong"] & \dots \\
\to H^k(U\cap V, U\cap V-(K\cap L)) \arrow[r] \arrow[d, "\mu_{K\cap L}\cap -"] & H^k(U,U-K)\oplus H^k(V,V-L) \arrow[r] \arrow[d, "\mu_K\cap - \oplus \mu_L \cap -"] & H^k(M,M-(K\cup L)) \arrow[r, "\delta^*"] \arrow[d, "\mu_{K\cup L}\cap -"] & \dots \\
\dots \to H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & \dots
\end{tikzcd} \quad (*)$$

MV in cohomology

MV-sequence in homology

Need to see that this diagram commutes.

Then, since $\{K\cap L \mid K \subseteq U, L \subseteq V \text{ compact}\}$ is cofinal in the set of all compact subsets of $U\cap V$ and similarly for $\{K\cup L\}$ in $U\cup V$,
we get the desired commutative ladder.

Commutativity in the shown part of $(*)$ follows from the functoriality of the cap product.

But it doesn't for

$$\begin{tikzcd}
H^k(M, M-(K\cup L)) \arrow[r] \arrow[d, "\mu_{K\cup L}\cap -"] & H^{k+1}(M, M-(K\cap L)) \cong H^{k+1}(U\cap V, U\cap V - K\cap L) \arrow[d, "\mu_{K\cap L}\cap -"] \\
H_{n-k}(M) \arrow[r, "\partial_*"] & H_{n-k-1}(U\cap V)
\end{tikzcd}$$

To show this commutativity, one has to use barycentric subdivision of $\mu_{K\cup L}$ and work on the chain level
$\rightsquigarrow$ reading assignment. $\square$

This concludes the proof of Poincaré duality.

Corollary. Any odd-dimensional manifold has zero Euler characteristic.

Pf: Suppose M is orientable and odd-dimensional.

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i(M;\mathbb{Q})$$

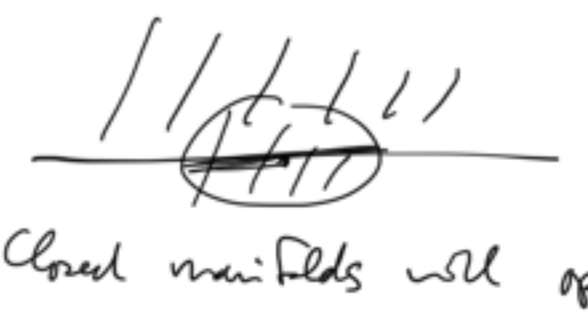
but $\dim H^i(M;\mathbb{Q}) = \dim H_{n-i}(M;\mathbb{Q}) = \dim H^{n-i}(M;\mathbb{Q})$

so they cancel out pairwise in the sum.

When M isn't orientable, use the orientation cover $\tilde{M}$

$$0 = \chi(\tilde{M}) = 2\chi(M).$$

Def. An n-dimensional *manifold with boundary* is a second countable Hausdorff space M s.t. every point has a neighborhood U homeomorphic with either $\mathbb{R}^n$ or $\mathbb{R}^n_{\geq 0} = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$.



Examples: Closed manifolds with open discs removed

$\overline{\mathbb{D}^n}$, intervals ...

Def. The *boundary* ∂M of a manifold with boundary is the subset $\{x \in M \mid \exists \text{ homeomorphism } \varphi: U \to \mathbb{R}^n_{\geq 0},\ \varphi(x) = 0\}$.

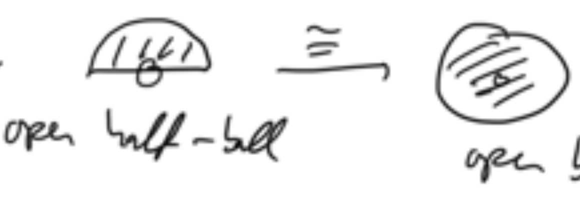
well-defined: points in ∂M do not have neighborhoods $\cong \mathbb{R}^n$

otherwise we'd have a homeomorphism

$U \to V$ $\quad V \subseteq \mathbb{R}^n$

$0 \in U \subseteq \mathbb{R}^n_{\geq 0}$ open

w.l.o.g.



not possible: removal of any point in an open ball results in a non-contractible space.

Remark: ∂M is an $(n-1)$-dim. manifold without boundary.

Cor.: If M is even dimensional, connected, and has odd Euler characteristic then M cannot be a boundary.

Pf: "Double of M"

$$\delta M = M \underset{\partial M}{\cup} M = M \sqcup M / \text{boundary points in the two copies are identified.}$$

$M =$   δM

This is a manifold without boundary.

$$\chi(\delta M) = 2\chi(M) - \chi(\partial M)$$

dim M odd: LHS $= 0 \Rightarrow \chi(\partial M)$ is even. $\square$

Examples: $\left.\begin{array}{ll} \mathbb{R}P^{2n} & \text{with } \chi = 2n+1 \\ \mathbb{C}P^{2n} & \text{with } \chi = 2n+1 \end{array}\right\}$ not boundaries

on the other hand, $\mathbb{R}P^1 = S^1$ is a boundary
$\mathbb{C}P^1 = S^2$ is a boundary
$\mathbb{R}P^3$ is a boundary (think!)