Lecture 14
Algebraic Topology

Cohomology with compact support

$H_c^*(X;R)$

Lemma. Let $\mathcal{K}$ be the poset of compact subsets of a space X.

Then $K \mapsto H^i(X, X-K)$ is a directed system of abelian groups.

and $H_c^i(X) \cong \operatorname{colim}_{K\in\mathcal{K}} H^i(X, X-K)$

Proof:

$$C^i(X, X-K) \xrightarrow{F_K} C_c^i(X) = \{\varphi \in C^i(X) \mid \exists K \subseteq X \text{ compact } \varphi(\sigma)=0 \text{ if } \sigma \in C_i(X-K)\}$$

$$\varphi : X^{\Delta^i} \to R, \quad \varphi|_{(X-K)^{\Delta^i}} = 0 \quad \mapsto \varphi$$

induces a unique map

$$\operatorname{colim}_K C^i(X, X-K) \to C_c^i(X)$$

by the universal property

and

$$\operatorname{colim}_K H^i(X, X-K) \to H_c^i(X)$$

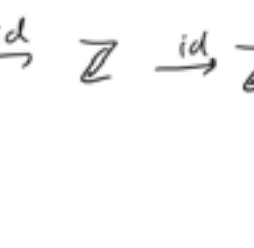
Surjective: every cycle in $C_c^i(X)$ vanishes on some K.

injective: if $[\varphi]=0 \in H_c^i(X)$ then $\varphi = \delta\psi$ for some $\psi \in C_c^{i-1}(X)$ that vanishes on some K.

$\Rightarrow \psi \in C^{i-1}(X, X-K)$. $\square$

Example. $H_c^*(\mathbb{R}^n) \cong \operatorname{colim}_{K \subseteq \mathbb{R}^n \text{ compact}} H^*(\mathbb{R}^n, \mathbb{R}^n - K)$

The subset $\{\overline{B_N(0)} \mid N \in \mathbb{N}\} \subseteq \mathcal{K}$ is cofinal



so $H_c^*(\mathbb{R}^n) \cong$

$$\cong \operatorname{colim}_{N\to\infty} H^*(\mathbb{R}^n, \mathbb{R}^n - \overline{B_N(0)}) = \tilde{H}^*(S^n) \cong \begin{cases} \mathbb{Z} & ; * = n \\ 0 & ; \text{o/w.} \end{cases}$$

$$\cong \operatorname{colim}(\mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \cdots) \text{ for } * = n$$

$$\cong \begin{cases} \mathbb{Z} & ; * = n \\ 0 & ; \text{o/w} \end{cases}$$

Functoriality. If $f: X \to Y$ is proper (preimages of compact sets are compact) then there is a functorially induced map

$$f^*: H_c^*(Y) \to H_c^*(X)$$

If $f: X \to Y$ is an open inclusion then there is a functorially induced map

$$f_*: H_c^*(X) \to H_c^*(Y)$$

Using the identification $H_c^*(X) \cong \tilde{H}^*(X^+)$ ← one-point compactification,

this map is induced by

$$f^+: Y^+ \to X^+$$
$$y \mapsto \begin{cases} f^{-1}(y) & ; y \in \operatorname{im}(f) \\ \infty & ; \text{o/w} \end{cases}$$

Thm (Poincaré duality for noncompact manifolds)

There is an isomorphism

$$D_M : H_c^k(M;R) \to H_{n-k}(M;R)$$

for R-oriented n-dimensional manifolds.

D_M is the colimit of the maps

$$\begin{aligned} H^k(M, M-K) &\longrightarrow H_n(M, M-K) \otimes H^k(M, M-K) \xrightarrow{\frown} H_{n-k}(M) \\ x &\longmapsto [M] \otimes x \longmapsto D_M(x) \end{aligned}$$

$[M]$: unique class restricts to the orientations μ_x on every $H_n(M, M-\{x\})$. (Theorem)

Lemma. If $M = U \cup V$, U, V open subsets, then there is a commutative-up-to sign ladder with exact rows

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H_c^k(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U) \oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus D_V"] & H_c^k(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] \arrow[dr, phantom, "-1"] & H^{k+1}(U\cap V) \arrow[d] \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U\cup V) \arrow[r] & H_{n-k-1}(U\cap V)
\end{tikzcd}$$

↑ Mayer-Vietoris sequence

Proof of Thm, assuming the Lemma.

Let $\mathcal{P}$ be the set of all open subsets of M for which D_U is an isomorphism.

To show: $M \in \mathcal{P}$.

Lemma $\Rightarrow$ $U, V, U\cap V \in \mathcal{P} \Rightarrow U \cup V \in \mathcal{P}$

Claim: If $U_1 \subset U_2 \subset U_3 \subset \cdots \subset \bigcup_{i=1}^\infty U_i = U$ open, $U_i \in \mathcal{P}\ \forall i \Rightarrow U \in \mathcal{P}$

proof:

$$\begin{tikzcd}[column sep=tiny]
& H_c^k(U_i) \arrow[dl, "D_{U_i}", "\cong"'] \arrow[d] \arrow[r, phantom, "\cong"] & \operatorname{colim}_{K\subseteq U_i} H^k(U_i, U_i - K) \arrow[r, phantom, "\overset{\text{excision}}{\cong}"] & \operatorname{colim}_{K\subseteq U_i} H^k(U, U-K) \arrow[d] \\
H_{n-k}(U_i) \arrow[d] & H_c^k(U_{i+1}) \arrow[dl, "\cong"] \arrow[d] \arrow[r, phantom, "\cong"] & \operatorname{colim}_{K\subseteq U_i} H^k(U_{i+1}, U_{i+1} - K) \arrow[r, phantom, "\cong"] & \operatorname{colim}_{K\subseteq U_{i+1}} H^k(U, U-K) \arrow[d] \\
H_{n-k}(U_{i+1}) \arrow[d] & & & \\
\operatorname{colim}_i H_{n-k}(U_i) = H_{n-k}(U) & & H_c^k(U) \arrow[ll, "D_U"', "\cong"] & = \operatorname{colim}_{K\subset U} H^k(U, U-K)
\end{tikzcd}$$

- If $U \cong \mathbb{R}^n$ then $U \in \mathcal{P}$.

$$\begin{tikzcd}
H_c^k(\mathbb{R}^n) \arrow[r, "D_{\mathbb{R}^n}"] & H_{n-k}(\mathbb{R}^n) \\
H^k(\mathbb{R}^n, \mathbb{R}^n - \overline{D}^n) \arrow[u, "\cong"] \arrow[r, "\cong"] & H_{n-k}(\mathbb{R}^n) \arrow[u, equal, "\cong"']
\end{tikzcd}$$

$H^k(\mathbb{R}^n, \mathbb{R}^n - \overline{D}^n) \cong \begin{cases} \mathbb{Z} & ; k = n \\ 0 & ; \text{o/w} \end{cases}$, $\quad x \mapsto [\mathbb{R}^n] \frown x$ where $[\mathbb{R}^n] \in H_n(\mathbb{R}^n, \mathbb{R}^n - D^n)$

- $\mathcal{P}$ = all open subsets of M:

pf: U open $\Rightarrow$ $U = \bigcup_{i=1}^\infty B_i$, B_i open ball contained in some chart $\mathbb{R}^n \subset M$

(M is second countable)

define $U_i = B_1 \cup \cdots B_i$

then ① $B_i \in \mathcal{P}$ because $B_i \cong \mathbb{R}^n$

② $U_i \in \mathcal{P}$ because they're finite unions

③ $U = \bigcup U_i$, $U_1 \subset U_2 \subset \cdots \in \mathcal{P}$