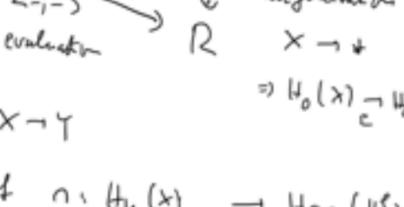


$$H_k(X; R) \otimes H^l(X; R) \rightarrow H_{k-l}(X; R)$$

Case  $k=l$ :

$$H_k(X; R) \otimes H^k(X; R) \xrightarrow{\cap} H_0(X; R)$$



Naturality: given  $f: X \rightarrow Y$

$$\begin{matrix} H_k(X; R) \otimes H^k(X; R) & \xrightarrow{\cap} & H_0(X; R) \\ \downarrow f_* & \otimes & \downarrow f_* \\ H_k(Y; R) \otimes H^k(Y; R) & \xrightarrow{\cap} & H_0(Y; R) \end{matrix}$$

Explicitly:  $f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi))$

Thm (Poincaré duality)

Let  $M$  be a compact,  $n$ -dimensional,  $R$ -oriented manifold with fundamental class  $[M] \in H_n(M; R)$

$$D: H^k(M; R) \rightarrow H_{n-k}(M; R)$$

$$\alpha \mapsto [M] \cap \alpha$$

is an isomorphism.

Equivalent formulation when either  $R$  is a field or  $H_*M$  is torsion free;

$$H^k(M; R) \otimes_R H^{n-k}(M; R) \xrightarrow{\cup} H^n(M; R) \cong \langle -, [M] \rangle_R$$

In other words, it induces

$$\text{an isomorphism } H^k(M; R) \rightarrow \text{Hom}_R(H^{n-k}(M; R), R)$$

Pf of equivalence:  $\langle x \cup y, [M] \rangle$

$$\begin{aligned} &= \varepsilon([M] \cap (x \cup y)) \\ &= \varepsilon((([M] \cap x) \cup y)) \quad [\text{kernel problem}] \\ &= \varepsilon(Dx \cap y) \\ &= \langle y, Dx \rangle \end{aligned}$$

Under the assumptions,  $H_{n-k}(M; R) \cong \text{Hom}(H^{n-k}(M), R)$

Def (Cohomology with compact support)

$$C_c^i(X) = \{ \varphi \in C^i(X) \mid \exists K \subseteq X \text{ compact } \varphi|_K = 0 \text{ if } \sigma \text{ is an } i\text{-simplex in } X-K \}$$

This is a sub-chain complex:  $\delta: C_c^i(X) \rightarrow C_c^{i+1}(X)$

Its homology  $H_c^i(X) = H^i(C_c^*(X))$  is cohomology with compact support.

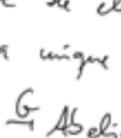
Rk:  $X$  compact  $\Rightarrow H_c^i(X) \cong H^i(X)$

will see:  $H_c^i(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}; & \text{if } i=n \\ 0; & \text{otherwise} \end{cases}$

Directed limits (colimit = "direct limit" = "inductive limit")

Def A partially ordered set  $I$  is directed if every two elements have an upper bound;

$$\forall \alpha, \beta \in I, \exists \gamma \in I: \gamma \geq \alpha, \gamma \geq \beta$$



This can be thought of as a category:

the objects are the elements of  $I$

and there is a unique morphism  $\alpha \rightarrow \beta$  iff  $\alpha \leq \beta$

A functor  $I \rightarrow \text{Abelian groups}$  is called a directed system of abelian groups.

Explicitly, it's a collection  $\{G_\alpha \mid \alpha \in I\}$  of ab. grps. and morphisms  $f_{\alpha\beta}: G_\alpha \rightarrow G_\beta$  whenever  $\alpha \leq \beta$

$$\text{s.t. } f_{\alpha\alpha} = \text{id}_{G_\alpha}, f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$$

Def The colimit  $\text{colim}_{\alpha \in I} G_\alpha = \text{colim } G$  is defined as

$$\bigoplus_{\alpha \in I} G_\alpha / \sim \quad \text{where for } g \in G_\alpha, g \sim f_{\alpha\beta}(g) \in G_\beta$$

More efficient description:

$$\text{colim}_{\alpha \in I} G_\alpha = \bigsqcup_{\alpha \in I} G_\alpha / \sim$$

where  $g_\alpha \in G_\alpha$  and  $g_\beta \in G_\beta$  are equivalent

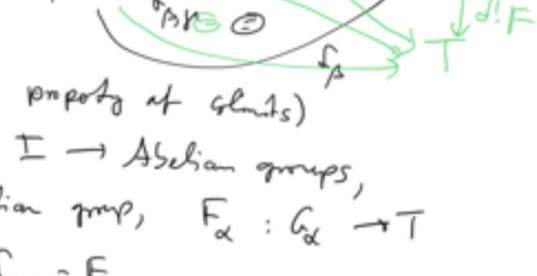
$$\text{if } \exists \gamma \geq \alpha, \beta \text{ s.t. } f_{\alpha\gamma}(g_\alpha) = f_{\beta\gamma}(g_\beta)$$

Addition is defined by  $[g_\alpha] + [g_\beta] = [f_{\alpha\gamma}(g_\alpha) + f_{\beta\gamma}(g_\beta)]$

Check: well-defined

There are homomorphisms  $f_\alpha: G_\alpha \rightarrow \text{colim } G_\alpha$

$$\text{s.t. } f_\beta \circ f_{\alpha\beta} = f_\alpha$$



Thm (universal property of colimits)

Given  $G: I \rightarrow \text{Abelian groups}$ ,

$T$  an abelian group,  $F_\alpha: G_\alpha \rightarrow T$

$$\text{s.t. } F_\beta \circ f_{\alpha\beta} = F_\alpha$$

Then there exists a unique map  $F: \text{colim } G_\alpha \rightarrow T$

$$\text{s.t. } F_\alpha = F \circ f_\alpha$$

Def  $J \subseteq I$ ,  $I$  directed set.

$J$  is called cofinal if  $\forall \alpha \in I \exists \beta \in J: \alpha \leq \beta$

Example:  $\mathbb{N}$  = natural numbers = totally ordered  $\Rightarrow$  directed

$$S \subseteq \mathbb{N} \text{ cofinal} \Leftrightarrow S \text{ infinite}$$

Lemma: Given  $G: I \rightarrow \text{Abelian groups}$ ,  $J \subseteq I$  cofinal

Then there is a natural isomorphism

$$\text{colim}_{\alpha \in I} G_\alpha \xrightarrow{\cong} \text{colim}_{\alpha \in J} G_\alpha$$

Proof:  $\text{colim}_{\alpha \in J} G_\alpha = \bigsqcup_{\alpha \in J} G_\alpha / \sim \rightarrow \bigsqcup_{\alpha \in I} G_\alpha / \sim = \text{colim}_{\alpha \in I} G_\alpha$

well-defined: if  $g_\alpha \sim g_\beta$  then  $\exists \gamma \in J \subseteq \text{colim } G_\alpha$

$$\text{s.t. } f_{\alpha\gamma}(g_\alpha) = f_{\beta\gamma}(g_\beta), \text{ so } g_\alpha \sim g_\beta \text{ in } \text{colim } G.$$

Surjective: given  $g_\alpha \in G_\alpha \quad \alpha \in I$ ,

$g_\alpha \sim f_{\alpha\beta}(g_\alpha)$  for  $\beta \in J, \beta \geq \alpha$ , which exists by the cofinality condition.

Injective: given  $[g_\alpha], [g_\beta] \in \text{colim } G_\alpha$

$$\text{s.t. } g_\alpha \sim g_\beta \text{ in } \text{colim } G.$$

Then  $\exists \gamma \geq \alpha, \beta, \gamma \in J$  s.t.

$$f_{\alpha\gamma}(g_\alpha) = f_{\beta\gamma}(g_\beta)$$

Choose  $\delta \in J, \delta \geq \gamma$

$$\Rightarrow f_{\alpha\delta}(g_\alpha) = f_{\beta\delta}(g_\beta)$$

$$\Rightarrow [g_\alpha] = [g_\beta] \text{ in } \text{colim } G_\alpha. \quad \square$$

Prop. Let  $X = \bigcup_{\alpha \in I} X_\alpha$ ,  $I$  directed set,

Assume every compact  $K \subseteq X$  is contained in some  $X_\alpha$  (e.g. nested open sets)

Then  $\text{colim}_{\alpha \in I} H_*(X_\alpha) \cong H_*(X)$ .

Pf: This is actually held - to be proved.