$$H_k(X;R) \otimes H^\ell(X;R) \to H_{k-\ell}(X;R)$$

Case $k=\ell$:

$$\begin{tikzcd} H_k(X;R) \otimes H^k(X;R) \arrow[r, "\cap"] \arrow[dr, "{\langle -,- \rangle}"', "\text{evaluation}"'] & H_0(X;R) \arrow[d, "\varepsilon\ \text{augmentation}"] \arrow[dl, phantom, "="] \\ & R \end{tikzcd}$$

$X \to *$

$\Rightarrow H_0(X) \xrightarrow{\varepsilon} H_0(*) = R$

Naturality: given $f: X \to Y$

easiest by thinking of $\cap$:

$$\begin{tikzcd} H_k(X) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "="] & \mathrm{Hom}(H^\ell X, H_{k-\ell} X) \arrow[d, "{\mathrm{Hom}(f^*, f_*)}"] \\ H_k(Y) \arrow[r] & \mathrm{Hom}(H^\ell Y; H_{k-\ell} Y) \end{tikzcd}$$

$g \mapsto f_* \circ g \circ f^*$

Explicitly: $f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi))$

<u>Thm</u> (Poincaré duality)

Let M be a compact, n-dimensional, R-oriented manifold with fundamental class $[M] \in H_n(M;R)$

Then
$$D: H^k(M;R) \longrightarrow H_{n-k}(M;R)$$
$$\alpha \longmapsto [M] \cap \alpha$$
is an isomorphism.

Equivalent formulation when either R is a field or $H_* M$ is torsion free;

$$\begin{tikzcd} H^k(M;R) \otimes_R H^{n-k}(M;R) \arrow[r, "\cup"] & H^n(M;R) \arrow[d, "\cong"', "{\langle -, [M] \rangle}"] \\ & R \end{tikzcd}$$

is a perfect pairing.

In other words, it induces an isomorphism $H^k(M;R) \longrightarrow \mathrm{Hom}_R(H^{n-k}(M,R), R)$

<u>pf of equivalence</u>: $\langle x \cup y, [M] \rangle$

$= \varepsilon([M] \cap (x \cup y))$

$= \varepsilon(([M] \cap x) \cap y)$ [homework problem]

$= \varepsilon(Dx \cap y)$

$= \langle y, Dx \rangle$

Under the assumptions, $H_{n-k}(M;R) \cong \mathrm{Hom}(H^{n-k}(M), R)$

//

<u>Def</u> (Cohomology with compact support)

$$C_c^i(X) = \{ \varphi \in C^i(X) \mid \exists K \subseteq X \text{ compact } \varphi(\sigma) = 0 \text{ if } \sigma \text{ is an } i\text{-simplex in } X-K \}$$

This is a sub-cochain complex: $\delta: C_c^i(X) \to C_c^{i+1}(X)$

Its homology $H_c^i(X) = H^i(C_c^*(X))$ is <u>cohomology with compact support</u>.

<u>Rk</u>: X compact $\Rightarrow H_c^i(X) = H^i(X)$

will see: $H_c^i(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}; & i=n \\ 0; & \text{otw.} \end{cases}$

<u>Directed colimits</u> (colimit = "direct limit" = "inductive limit")

<u>Def</u>. A partially ordered set I is <u>directed</u> if every two elements have an upper bound;

$\forall \alpha, \beta \in I, \exists \gamma \in I: \gamma \geq \alpha, \gamma \geq \beta$

$$\begin{tikzcd}[row sep=small] \alpha \arrow[dr] & \\ & \gamma \\ \beta \arrow[ur] & \end{tikzcd}$$

This can be thought of as a category:

the objects are the elements of I

and there is a unique morphism $\alpha \to \beta$ iff $\alpha \leq \beta$,

A functor $I \xrightarrow{G}$ Abelian groups is called a directed system of abelian groups.

Explicitly, it's a collection $\{G_\alpha \mid \alpha \in I\}$ of ab. grps. and morphisms $f_{\alpha\beta}: G_\alpha \to G_\beta$ whenever $\alpha \leq \beta$

s.t. $f_{\alpha\alpha} = \mathrm{id}_{G_\alpha}$, $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$.

<u>Def</u> The <u>colimit</u> $\operatorname{colim}_{\alpha \in I} G_\alpha = \operatorname{colim} G$ is defined as

$$\bigoplus_{\alpha \in I} G_\alpha / \sim \quad \text{where for } g \in G_\alpha,\ g \sim f_{\alpha,\beta}(g) \in G_\beta.$$

More efficient description:

$$\operatorname{colim}_{\alpha \in I} G_\alpha = \coprod_{\alpha \in I} G_\alpha / \sim$$

where $g_\alpha \in G_\alpha$ and $g_\beta \in G_\beta$ are equivalent if $\exists \gamma \geq \alpha, \beta$ s.t. $f_{\alpha,\gamma}(g_\alpha) = f_{\beta,\gamma}(g_\beta)$

Addition is defined by $[g_\alpha] + [g_\beta] = [f_{\alpha,\gamma}(g_\alpha) + f_{\beta,\gamma}(g_\beta)]$

<u>Check</u>: well-defined

There are homomorphism $f_\alpha: G_\alpha \longrightarrow \operatorname{colim}_{\alpha \in I} G_\alpha$

s.t. $f_\beta \circ f_{\alpha\beta} = f_\alpha$

$$\begin{tikzcd} G_\alpha \arrow[d, "f_{\alpha\beta}"'] \arrow[dr, "f_{\alpha\gamma}"] \arrow[rr, bend left, "f_\alpha"] & & \operatorname{colim}_{\alpha \in I} G_\alpha \arrow[d, "\exists! F"] \\ G_\beta \arrow[r, "f_{\beta\gamma}"'] \arrow[urr, bend right, "f_\beta"'] \arrow[rr, bend right] & G_\gamma \arrow[ur, "f_\gamma"] \arrow[r] & T \end{tikzcd}$$

<u>Thm</u> (universal property of colimits)

Given $G: I \longrightarrow$ Abelian groups,

T an abelian group, $F_\alpha: G_\alpha \to T$

s.t. $F_\beta \circ f_{\alpha,\beta} = F_\alpha$

Then there exists a unique map $F: \operatorname{colim}_{\alpha \in I} G_\alpha \to T$

s.t. $F_\alpha = F \circ f_\alpha$

<u>Def</u> $J \subset I$, I directed set.

J is called <u>cofinal</u> if $\forall \alpha \in I\ \exists \beta \in J: \alpha \leq \beta$

<u>Example</u>; $\mathbb{N}$ = natural numbers = totally ordered $\Rightarrow$ directed

$S \subseteq \mathbb{N}$ cofinal $\Leftrightarrow S$ infinite

<u>Lemma</u>: Given $G: I \to$ Abelian groups, $J \subseteq I$ cofinal

Then there is a natural isomorphism

$$\operatorname{colim}_{\alpha \in I} G_\alpha \longleftarrow \operatorname{colim}_{\alpha \in J} G_\alpha.$$

<u>Proof</u>:
$$\operatorname{colim}_{\alpha \in J} G_\alpha = \coprod_{\alpha \in J} G_\alpha / \sim \longrightarrow \coprod_{\alpha \in I} G_\alpha / \sim = \operatorname{colim}_{\alpha \in I} G_\alpha$$
$$g_\alpha \longmapsto g_\alpha$$

well-defined: if $g_\alpha \sim g_\beta$ then $\exists \gamma \in J$ s.t. $f_{\alpha,\gamma}(g_\alpha) = f_{\alpha,\gamma}(g_\beta)$, so $g_\alpha \sim g_\beta$ in $\operatorname{colim} G$.

$\in \operatorname{colim} G|_J$

<u>Surjective</u>: given $g_\alpha \in G_\alpha$, $\alpha \in I$,

$g_\alpha \sim f_{\alpha,\beta}(g_\alpha)$ for $\beta \in J$, $\beta \geq \alpha$, which exists by the cofinality condition.

<u>injective</u>: given $[g_\alpha], [g_\beta] \in \operatorname{colim} G|_J$

s.t. $g_\alpha \sim g_\beta$ in $\operatorname{colim} G$.

Then $\exists \gamma \geq \alpha, \beta$, $\gamma \in I$ s.t.

$f_{\alpha,\gamma}(g_\alpha) = f_{\beta,\gamma}(g_\beta)$

Choose $\delta \in J$, $\delta \geq \gamma$

$\Rightarrow f_{\alpha,\delta}(g_\alpha) = f_{\beta,\delta}(g_\beta)$

$\Rightarrow [g_\alpha] = [g_\beta]$ in $\operatorname{colim} G|_J$. $\square$

<u>Prop</u>. Let $X = \bigcup_{\alpha \in I} X_\alpha$, I directed set,

Assume every compact $K \subseteq X$ is contained in some X_α (e.g. nested open sets)

Then $\operatorname{colim}_{\alpha \in I} H_*(X_\alpha) \cong H_*(X)$.

pf: This actually holds ...