Algebraic Topology
Lecture 13

Recap: $J : H_n(M, M-A; R) \longrightarrow \Gamma_c(A; R)$

$c \longmapsto (x \mapsto \mathrm{Res}_x\, c)$

Thm. ① $H_i(M, M-A; R) = 0$ for $i > n$

② J is an isomorphism. $\square$

(with any coefficients)

Corollary. M n-dimensional manifold, $A \subseteq M$ closed subset.
A connected

$$H_n(M, M-A; R) = \begin{cases} 0; & A \text{ noncompact} \\ R; & A \text{ compact, } M \text{ } R\text{-orientable along } A \\ {}_2R; & A \text{ compact, } M \text{ not } R\text{-or. along } A \end{cases}$$

where ${}_2R = \{x \in R \mid 2x = 0\}$.

proof. $s \in \Gamma_c(A) \cong H_n(M, M-A; R)$ is determined by its value on a single point.

- A noncompact, connected, $s \neq 0 \Rightarrow \exists x \in A : s(x) \neq 0$
$\Rightarrow \forall x \in A \;\; s(x) \neq 0 \Rightarrow \mathrm{supp}(s) = A$ not compact.

Now suppose that A is compact.

$$\begin{array}{ccc} H_n(M, M-A) & \underset{J}{\cong} & \Gamma(A) \\ \downarrow \mathrm{res} & & \downarrow b \text{ injective.} \\ H_n(M, M-\{x\}) & \cong & \Gamma(\{x\}) \cong R \end{array}$$

If M is orientable along $A \Rightarrow \exists s \in \Gamma(A)$ s.t. $b(s)$ is a generator. $\Rightarrow b$ isomorphism.

M not orientable along A

A section $s \in \Gamma(A; R)$ is given by a map

$\tilde{M}|_A \xrightarrow{\lambda} R$ s.t. $\lambda \circ t = -\lambda$

Deck transformation of $\tilde{M}$ flipping the two sheets

But $\tilde{M}|_A$ is connected

$\Rightarrow \lambda = $ constant, $\lambda \circ t = \lambda$

$\Rightarrow \lambda = -\lambda \Rightarrow \lambda \in {}_2R$. $\square$

In particular, if $A = M$, we have

$$H_n(M; R) = \begin{cases} 0 & M \text{ noncompact} \\ R & M \text{ orientable} \\ {}_2R & M \text{ nonorientable} \end{cases} \Big\} \text{ compact}$$

Thus a generator of $H_n(M; \mathbb{Z})$ determines an orientation of M and vice versa. Such a generator is called a fundamental class and denoted by $[M]$.

Cor. M compact, connected n-manifold

M orientable $\Rightarrow H_{n-1}(M; \mathbb{Z})$ is torsion free

M nonorientable $\Rightarrow H_{n-1}(M; \mathbb{Z})_{\mathrm{tor}} = \mathbb{Z}/2\mathbb{Z}$

proof: Universal coefficient sequence;

$$0 \to H_n(M) \otimes \mathbb{Z}/k\mathbb{Z} \to H_n(M; \mathbb{Z}/k\mathbb{Z}) \to \mathrm{Tor}(H_{n-1}(M); \mathbb{Z}/k\mathbb{Z}) \to 0.$$

orientable:

$\mathbb{Z}/k\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/k\mathbb{Z}$

$\Rightarrow \mathrm{Tor}(H_{n-1}(M); \mathbb{Z}/k\mathbb{Z}) = 0$ for all k.

$\Rightarrow H_{n-1}(M)$ torsion free.

nonorientable,

$0 \qquad \begin{cases} 0; & k \text{ odd} \\ \mathbb{Z}/2\mathbb{Z}; & k \text{ even} \end{cases}$

$$\Rightarrow \mathrm{Tor}(H_{n-1}(M); \mathbb{Z}/k\mathbb{Z}) = \begin{cases} 0; & k \text{ odd} \\ \mathbb{Z}/2\mathbb{Z}; & k \text{ even.} \end{cases}$$

$\Rightarrow H_{n-1}(M)_{\mathrm{tor}} = \mathbb{Z}/2\mathbb{Z}$ $\square$

The cap product

Def: $\cap : C_k(X; R) \otimes C^\ell(X, R) \to C_{k-\ell}(X; R)$ $(k \geq \ell)$

$\sigma \otimes \varphi \mapsto (\sigma|_{[v_0 \ldots v_\ell]}) \cdot \sigma|_{[v_\ell \ldots v_k]}$

To see that this induces a map in H_* & H^*, we compute:

$$\partial(\sigma \cap \varphi) = \varphi(\sigma|_{[v_0 \ldots v_\ell]}) \cdot \sum_{i=\ell}^{k} (-1)^{i-\ell} \sigma|_{[v_\ell \ldots \hat{v}_i \ldots v_k]}$$

$$= (-1)^\ell \cdot \sum_{i=\ell}^{k} (-1)^i \varphi(\sigma|_{[v_0 \ldots v_\ell]}) \cdot \sigma|_{[v_\ell \ldots \hat{v}_i \ldots v_k]}$$

$$(\partial\sigma) \cap \varphi = \sum_{i=0}^{\ell} (-1)^i \varphi(\sigma|_{[v_0 \ldots \hat{v}_i \ldots v_{\ell+1}]}) \cdot \sigma|_{[v_{\ell+1} \ldots v_k]}$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \varphi(\sigma|_{[v_0 \ldots v_\ell]}) \sigma|_{[v_\ell \ldots \hat{v}_i \ldots v_k]}$$

$$\sigma \cap (\delta\varphi) = \sum_{i=0}^{\ell+1} (-1)^i \varphi(\sigma|_{[v_0 \ldots \hat{v}_i \ldots v_{\ell+1}]}) \sigma|_{[v_{\ell+1} \ldots v_k]}$$

So $(\partial\sigma) \cap \varphi - \sigma \cap (\delta\varphi)$

$$= \overbrace{-(-1)^{\ell+1}}^{= (-1)^\ell} \varphi(\sigma|_{[v_0 \ldots v_\ell]}) \sigma|_{[v_{\ell+1} \ldots v_k]}$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \varphi(\sigma|_{[v_0 \ldots v_\ell]}) \sigma|_{[v_\ell \ldots \hat{v}_i \ldots v_k]}$$
$$= (-1)^\ell \partial(\sigma \cap \varphi)$$

So $\partial(\sigma \cap \varphi) = (-1)^\ell \big((\partial\sigma) \cap \varphi - \sigma \cap (\delta\varphi)\big)$.

if σ is a cycle and φ is a cocycle then $\sigma \cap \varphi$ is a cycle

if $\sigma = \partial\bar{\sigma}$ is a boundary and φ is a cocycle,
$\partial(\bar{\sigma} \cap \varphi) = \pm\, \sigma \cap \varphi - 0$, so $\sigma \cap \varphi$ is a boundary.

if $\varphi = \delta\bar{\varphi}$ is a coboundary and σ a cycle, $\sigma \cap \varphi$ is a boundary.

Thus: There is an induced cap product

$$\cap : H_k(X; R) \otimes H^\ell(X; R) \to H_{k-\ell}(X; R)$$

relative versions:

- $H_k(X, A; R) \otimes H^\ell(X; R) \xrightarrow{\cap} H_{k-\ell}(X, A; R)$
- $H_k(X, A; R) \otimes H^\ell(X, A; R) \xrightarrow{\cap} H$