Lecture 12 part 2

Prop. Let $A \subseteq M$ be a closed subset. Then there is a homomorphism

$$J : H_n(M, M-A) \longrightarrow \Gamma_c(A) \quad \text{(any coefficients)}$$

$\uparrow$ sections of $\mathcal{H}_n(M)$ with compact support

$$\alpha \longmapsto (x \mapsto \mathrm{res}^A_x \alpha)$$

Proof: Let α be represented by a cycle $c \in C_n(M)$

$K = \mathrm{supp}(c) \subseteq M$ compact

$c = \sum_{i=1}^{n} \lambda_i \sigma_i$ for some $\sigma_i : \Delta^n \to M$, $\lambda_i \neq 0$.

$\mathrm{supp}(c) := \bigcup_{i=1}^{n} \mathrm{im}(\sigma_i) \subseteq M$

For $x \in A - K$

$$\begin{tikzcd} C_n(K) \arrow[r] & C_n(M) \arrow[r] & C_n(M,K) \arrow[r] & C_n(M, M-\{x\}) \\ \tilde{c} \arrow[r, dashed, mapsto] \arrow[rr, bend right, "0 \text{ in homology.}"'] & c \arrow[rr, mapsto, "\mathrm{res}"] & & \mathrm{res}_x c \end{tikzcd}$$

$\mathrm{res}_x c$ represents $\mathrm{res}_x \alpha = 0$.

So $\mathrm{supp}(J(\alpha)) \subseteq A \cap K$, which is compact $\square$

Thm $A \subseteq M$ closed subset.

① $H_i(M, M-A) = 0$ when $i > n$

② $J : H_n(M, M-A) \to \Gamma_c(A)$ is an isomorphism.

proof. "bootstrapping"

(I) ① & ② hold when $A = \varphi^{-1}(\mathbb{D}^n)$

for $\mathbb{D}^n \subseteq \mathbb{R}^n \xleftarrow[\cong]{\varphi} U \subseteq M$

pf: $x = \varphi^{-1}(0)$

$$\begin{tikzcd} H_i(M, M-A) \arrow[d, "\mathrm{res}"] & H_i(U, U-A) \arrow[l, "\cong"', "\mathrm{exc}"] \arrow[r, "\varphi_*", "\cong"'] \arrow[d, "\mathrm{res}"] & H_i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{D}^n) \arrow[d, "\mathrm{res}", "\cong"'] \\ H_i(M, M-\{x\}) & H_i(U, U-\{x\}) \arrow[l, "\cong"', "\mathrm{exc}"] \arrow[r, "\varphi_*", "\cong"'] & H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \end{tikzcd}$$

$$H_i(\mathbb{R}^n, \mathbb{R}^n-\{0\}) \cong \begin{cases} 0 ; & i > n \\ \mathbb{Z} ; & i = n \end{cases}$$

$\Rightarrow$ ① follows

$\Rightarrow$ ② follows because a section over a connected set is determined by a single value.

(II) If A, B are closed subsets and A, B, and $A \cap B$ satisfy ① + ② then so does $A \cup B$.

Proof: Mayer-Vietoris sequence

$$\begin{tikzcd} H_{n+1}(M, M-(A \cap B)) \arrow[r, "\cong"'] \arrow[d] & 0 \arrow[d] \\ H_n(M, M-(A \cup B)) \arrow[r, "J"] \arrow[d] & \Gamma_c(A \cup B) \arrow[d] \\ H_n(M, M-A) \oplus H_n(M, M-B) \arrow[r, "J", "\cong"'] \arrow[d] & \Gamma_c(A) \oplus \Gamma_c(B) \arrow[d] \\ H_n(M, M-(A \cap B)) \arrow[r, "J", "\cong"'] & \Gamma_c(A \cap B) \end{tikzcd}$$

proves nd case of (1)

$\Rightarrow \cong$ proves (2)

exact columns

(III) If $A = K_1 \cup \dots \cup K_r \subseteq \mathbb{R}^n \xleftarrow[\varphi]{\cong} U \supseteq M$

K_i compact & convex

Then ① + ② hold for A (uses (I) and (II))

(IV) Let B be any compact subset $\subseteq \mathbb{R}^n \xleftarrow[\cong]{\varphi} U \supseteq M$

$A = \varphi^{-1}(B)$. Then ① + ② hold for A.

pf: Write $A = \bigcap_{i=1}^{\infty} K_i$, $K_1 \supset K_2 \supset K_3 \supset \dots$

K_i are finite unions of compact convex sets as in (III)

(Idea: cover A by finitely many closed ε-balls, letting $\varepsilon \to 0$)

Then

$$\begin{tikzcd} \mathrm{colim}\, H_n(M, M-K_j) \arrow[r, "\mathrm{res}", "\cong"'] \arrow[d, "\cong \text{ by (III)}"] & H_n(M, M-A) \arrow[d, "J"] \\ \mathrm{colim}\, \Gamma_c(K_j) \arrow[r, "\mathrm{res}", "\cong"'] & \Gamma_c(A) \end{tikzcd}$$

[**Aside:** Colimits of abelian groups

Given $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$ a sequence of abelian groups, define

$$\mathrm{colim}\, A_i = \bigoplus_{i=1}^{\infty} A_i \Big/ a_i \sim f_i(a_i).$$

[**Example:** If $A_1 \subseteq A_2 \subseteq A_3$ then $\mathrm{colim}\, A_i = \bigcup A_i$

Property: there is a canonical map

$$g_i : A_i \to \mathrm{colim}\, A_i$$

- For every $x \in \mathrm{colim}\, A_i$ there is an i and $x_i \in A_i$ s.t. $g_i(x_i) = x$
- For every $x_i \in A_i$, $x_j \in A_j$ s.t. $g_i(x_i) = g_j(x_j)$ there is a k s.t. $f_{k-1} \circ \dots \circ f_i(x_i) = f_{k-1} \circ \dots \circ f_j(x_j)$. $(k \geq i,j)$]

Top row is an isomorphism by additivity of homology

Exercise: Show that if $X_1 \subset X_2 \subset \dots$ are top spaces (closed inclusions) then $H_n(\bigcup X_i) = \mathrm{colim}\, H_n(X_i)$

Bottom row: homework.

(V) ① & ② hold for arbitrary compact A.

pf: A is a finite union of compact subsets contained in coordinate charts (use (II))

(VI) ① & ② hold when A is an (infinite) disjoint union of compact sets

pf: additivity of H_* and of Γ

(VII) A = any closed subset.

M = locally compact, countable basis

$\Rightarrow M = \bigcup K_i$ $\quad K_1 \subset K_2 \subset K_3$ compact subsets

$K_i \subseteq K^{\circ}_{i+1}$

$A_0 = K_1 \cap A$

$A_i = (K_i - K^{\circ}_{i-1}) \cap A$

$B = \coprod_{i \text{ even}} A_i$

$C = \coprod_{i \text{ odd}} A_i$

B, C are of the form (VI), as is $B \cap C$

$\Rightarrow$ (II) ① & ② hold for $B \cup C = A$ $\square$