# Lecture 12

## Algebraic Topology

following tom Dieck, "Algebraic Topology"

**Def.** An <u>orientation of $M$</u> consists of local orientations
$$\mu_x \in H_n(M, M-\{x\}) \cong \mathbb{Z} \quad \text{for each } x\in M$$
such that for each $x \in M$ there is a neighborhood $\mathcal{U} \ni x$, $\mu_{\mathcal U} \in H_n(M, M-\mathcal{U})$ st.
$$\mu_{\mathcal U}|_y = \mu_y \text{ for all } y \in \mathcal{U}$$
Here $\mu_{\mathcal U}|_y = \mathrm{Res}^{\mathcal U}_y \mu_{\mathcal U}$ = image of $\mu_{\mathcal U}$ under
$$H_n(M, M-\mathcal{U}) \xrightarrow{\text{res}} H_n(M, M-\{y\}).$$
will see: for <u>compact</u> $M$, this is equivalent to last week's definition.

**Lemma** Let $x \in M$, $W \ni x$ neighborhood.
Then there exists a possibly smaller neighborhood $\mathcal U$, $x \in \mathcal{U} \subseteq W$ s.t.
$$H_n(M, M-\mathcal{U}) \xrightarrow{\mathrm{Res}^{\mathcal U}_y} H_n(M, M-\{y\})$$
is an isomorphism for all $y \in \mathcal{U}$.

**proof:** Choose a homeomorphism $\varphi: V \to \mathbb{R}^n$, $V \subseteq W$, $x \in V$. Let $\mathcal{U} = \varphi^{-1}(\mathring{D}^n) \subseteq V$.

Then
$$\begin{tikzcd}
H_n(M, M-\mathcal{U}) \arrow[r, "\text{exc}", "\cong"'] \arrow[d, "\text{res}^{\mathcal U}_y"] & H_n(V, V-\mathcal{U}) \arrow[r, "\varphi", "\cong"'] \arrow[d, "\text{res}^{\mathcal U}_y"] & H_n(\mathbb{R}^n, \mathbb{R}^n-\mathring{D}^n) \arrow[d, "\text{res}^{\mathring{D}^n}_{\varphi(y)}", "\cong"'] \\
H_n(M, M-\{y\}) \arrow[r, "\text{exc}", "\cong"'] & H_n(V, V-\{y\}) \arrow[r, "\varphi", "\cong"'] & H_n(\mathbb{R}^n, \mathbb{R}^n-\{\varphi(y)\})
\end{tikzcd}$$
$\Rightarrow \text{res}^{\mathcal U}_y$ is an iso. □

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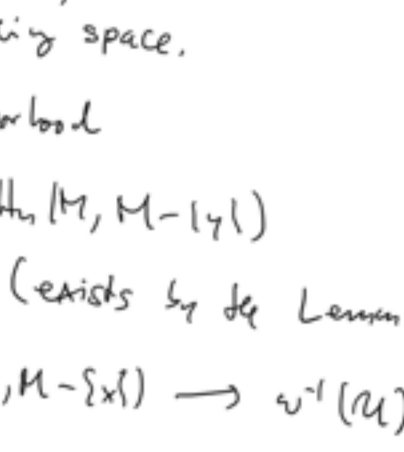
**Def.** $\mathcal{H}_n(M; R) := \coprod_{x\in M} H_n(M, M-\{x\}; R) \cong \coprod_{x \in M} \cancel{\mathbb{Z}}\ R$

$R$ abelian group

$$\begin{array}{cc} \downarrow \omega & \overset{*}{\downarrow} \\ M & x \end{array}$$




We want to give it a topology so that it becomes a covering space.

Pick $x \in M$, $\mathcal{U} \ni x$ neighborhood s.t. $H_n(M, M-\mathcal{U}) \xrightarrow{\text{res}^{\mathcal U}_y} H_n(M, M-\{y\})$ is an iso for all $y \in \mathcal{U}$ (exists by the Lemma)

define
$$\varphi_{x,\mathcal U}: \mathcal{U} \times H_n(M, M-\{x\}) \longrightarrow \omega^{-1}(\mathcal{U})$$
$$(y, a) \longmapsto \text{res}^{\mathcal U}_y \left(\text{res}^{\mathcal U}_x\right)^{-1}(a)$$
Topologize $\mathcal{H}_n(M)$ s.t. $\varphi_{x,\mathcal U}$ is a homeomorphism onto an open subset of $\mathcal{H}_n(M)$.

Is this well-defined, i.e. independent of the choice of $x, \mathcal{U}$?

Need to see: on $\mathcal{U} \cap V$, $x \in \mathcal{U}$, $y \in V$
$$\varphi^{-1}_{y,V} \circ \varphi_{x,\mathcal U}: \mathcal{U} \cap V \times H_n(M, M-\{x\}) \to \mathcal{U} \cap V \times H_n(M, M-\{y\})$$
is continuous.

Given $z \in \mathcal{U} \cap V$, choose (by the Lemma) $W$, $z \in W \subseteq \mathcal{U} \cap V$ s.t. $H_n(M, M-W) \xrightarrow{\cong} H_n(M, M-\{w\})$ for all $w \in W$

Then we have:
$$\begin{tikzcd}
H_n(M, M-\{x\}) & H_n(M, M-\mathcal{U}) \arrow[l, "\text{res}"'] \arrow[r, "\text{res}"] \arrow[d, "\text{res}"] & H_n(M, M-\{w\}) \\
& H_n(M, M-W) & H_n(M, M-V) \arrow[l, "\text{res}"'] \arrow[u, "\text{res}"'] \arrow[d, "\text{res}"] \\
& & H_n(M, M-\{y\})
\end{tikzcd}$$
$\Rightarrow \varphi^{-1}_{y,V} \circ \varphi_{x,\mathcal U}$ is independent of $w$

$\Rightarrow$ continuous. □

<u>Remark</u>: $\mathcal{H}_n(M) \to M$ is a covering space with fiber $\mathbb{Z}$. (R)

**Def.** For $A \subseteq M$, define
$$\Gamma(A) = \{ f: A \xrightarrow{\text{continuous}} \omega^{-1}(A) \mid \omega \circ f = id_A \}$$
the abelian group of sections of $\omega$ over $A$
$$\Gamma_c(A) < \Gamma(A)$$
$$\Gamma_c(A) = \{ f \in \Gamma(A) \mid \text{supp}(f) \text{ is compact} \}$$
$$\text{supp}(f) = \overline{\{x \in M \mid f(x) \neq 0\}}.$$

**Prop.** For $z \in H_n(M, M-\mathcal{U})$, define for $y \in \mathcal{U}$
$$f(y) = \mathrm{Res}^{\mathcal U}_y z$$
Then $f \in \Gamma(\mathcal{U})$. □

**Def** Let $R$ be an abelian group (often a ring)
An <u>$R$-orientation</u> of $M$ along $A$ is a section $s \in \Gamma(A; R) = \{ f: A \to \mathcal{H}_n(M; R) \mid \omega \circ f = id_A \}$ s.t. $s(a) \in H_n(M, M-\{a\})$ is a generator for all $a$.
An <u>$R$-orientation of $M$</u> is an $R$-orientation of $M$ along $M$.

The <u>orientation covering</u> $\tilde{M} \to M$ is the subset $\tilde{M} \subseteq \mathcal{H}_n(M; \mathbb{Z})$ of generators in all fibers, a double covering of $M$.

**Proposition.** TFAE:

(1) $M$ is orientable

(2) $M$ is orientable along all compact subsets

(3) $\tilde{M} \to M$ is a trivial covering, i.e. $\tilde M = M \sqcup M$

(4) $\omega$ is a trivial $\mathbb{Z}$-covering, i.e. $\mathcal{H}_n(M) \cong M \times \mathbb{Z}$.

<u>pf</u> (1) $\Rightarrow$ (2) clear.

(2) $\Rightarrow$ (3) w.l.o.g. suppose that $M$ is connected.

Then $\tilde{M} \to M$ is trivial iff $\tilde M$ is <u>not</u> connected.

Suppose $\tilde{M} \to M$ is nontrivial. Given $x \in M$, choose a path $\gamma: [0,1] \to \tilde{M}$ s.t. $\gamma(0), \gamma(1)$ are the two preimages of $x$ in $\tilde{M}$.

$\text{im}(p \circ \gamma) = S \subseteq M$ is compact, and the covering is nontrivial over $S$. This contradicts (2).

(3) $\Rightarrow$ (4) $M \times \mathbb{Z} \to \mathcal{H}_n(M)$ is a trivialization
$$(m, t) \mapsto t \cdot s(m)$$
if $s: M \to \tilde{M}$ is a section.

(4) $\Rightarrow$ (1) is clear. □

<u>Ex.</u>: $\widetilde{\mathbb{RP}^n} = S^n$

$\widetilde{\mathbb{CP}^n} = \mathbb{CP}^n \sqcup \mathbb{CP}^n$

$M$ = Möbius band $= [0,1] \times \mathbb{R} /_{(0,t) \sim (1,-t)}$

$\tilde{M} = S^1 \times \mathbb{R}$