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This test is useful when we have something that looks like n^n

Ex: $\sum_{n=1}^{\infty}\left(\frac{2n}{5+3n}\right)^n$ converges since $\lim_{n\to\infty}\left(\left(\frac{2n}{5+3n}\right)^n\right)^{1/n} = \lim_{n\to\infty}\frac{2n}{5+3n} = \frac{2}{3} < 1$

$\longrightarrow$

See next page

Taylor & Maclaurin Series

Recall if we had a "nice" function $f(x)$ then we can estimate it with Taylor polynomials.

That is,

$$f(x) \approx P_n(x) := \sum_{k=0}^{n} \frac{f^{k}(c)}{k!}(x-c)^k$$

Now that we know about series we can look at what happens as we let our approximation:

Definition If $f(x)$ has derivatives of all orders at c ($f^{k}(c)$ exists for all k) then we define

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$ to be the

Taylor Series of f about c.

If $c=0$ we say Maclaurin series

Alternating Series Test

Suppose $\{a_n\}$ is a sequence which satisfies

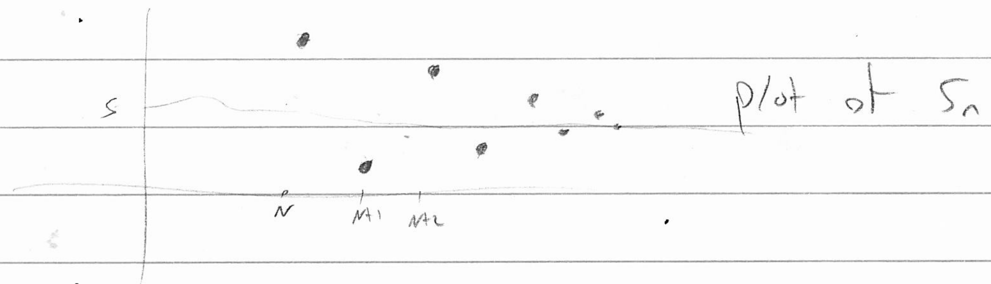
a) $a_n a_{n+1} < 0$ for $n \geq N$ (ultimately alternating)

b) $|a_n| < |a_{n+1}|$ for $n \geq N$ (ultimately decreasing in size)

c) $\lim_{n\to\infty} a_n = 0$

Then, $\sum_{n=1}^{\infty} a_n$ converges.

"Proof" by picture



Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges! But to what?

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Definition A function f is analytic at c if f has a Taylor series at c that converges to f(x) in an open interval. If f is analytic at each point of an open interval, then we say it is analytic on that interval.

Comment: if f is an analytic on an interval I we will just write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k, \quad \text{for } x \in I$$

Caution: The caveat that $x \in I$ is crucial. It may be that this series does not converge for x values not in I and so it wouldn't make sense to equal them.

This interval is called the integral of convergen and The length of this interval is called the radius of convergence.

Ex: Find the taylor series for e^x about c and find it's interval of convergence

We know that if $f(x)=e^x$ the $f^{(k)}(x)=e^x$ so $f^{k}(c)=e^c$ and so the taylor series is

$$\sum_{k=0}^{\infty} \frac{e^c}{k!}(x-c)^k.$$

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Use Ratio test to find for what x this converges: i.e. $a_n = \frac{e^c}{n!}(x-c)^n$

So $\frac{a_{n+1}}{a_n} = \frac{e^c (x-c)^{n+1}/(n+1)!}{e^c (x-c)^n / n!} = \frac{x-c}{n+1} \to 0$ as $n \to \infty$ for all x

So $\sum_{k=0}^{\infty} \frac{e^c}{k!}(x-c)^k$ converges for all x.

Thus we write

$$e^x = \sum_{k=0}^{\infty} \frac{e^c}{k!}(x-c)^k$$

More often though we set $c=0$ and write

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(In my opinion this is how e^x should be thought of)

This also gives us a way to write down e:

$$e = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$$

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Theorem (Uniqueness of Taylor series)

Suppose there exists a sequence $\{a_n\}$ such that
$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$
for x in some interval containg c, then
$$a_n = \frac{f^{(n)}(c)}{n!}$$

Proof Indeed, $f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$
$f''(x) = 2a_2 + 6a_3(x-c) + 12(x-c)^2 + \cdots$
$f'''(x) = 6a_3 + 24a_4(x-c) + \cdots$

And so $f(c) = a_0$, $f'(c) = a_1$, $f''(c) = 2a_2$, $f'''(c) = 3a_3$, $\cdots$

$\blacksquare$

We have seen that ~~the~~ taylor polynomials for $\sin x$ & $\cos x$ are
$$\sin(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \cos x \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

And so
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

And we can show this converges for all x in much the same way we showed it for e^x.

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Further, we have seen through geometric series that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } -1 < x < 1$$

And that this is <u>not</u> true for any x outside this range.

Find the Taylor series of $\ln(1-x)$.

Have that $\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n$ for $-1<x<1$

So: $\ln(1-x) = \int -\sum_{n=0}^{\infty} x^n \, dx = -\sum_{n=0}^{\infty} \int x^n \, dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$= -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $-1<x<1$

But does it converge for more x's ?

<u>Ratio Test:</u> $\lim_{n\to\infty} \frac{x^{n+1}}{n+1} / \frac{x^n}{n} = \lim_{n\to\infty} x \cdot \frac{n}{n+1} = x$

So diverges for $|x| > 1$ But inconclusive for $x = \pm 1$

So check $x = \pm 1$:

$x=1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $x=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ <u>converges</u>

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So, we may conclude that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for} \quad -1 \le x < 1$$

In particular, we then have that

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots = \ln(1-(-1)) = \ln(2).$$

Similarily, since

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $-1 < -x^2 < 1$

we get $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $-1 < x < 1$

But can it converge for larger intervals?

Ratio test: $\lim_{n \to \infty} \frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1} \Big/ \frac{(-1)^n x^{2n+1}}{2n+1} = x^2$

So diverges for $|x| > 1$ check $x = \pm 1$

$x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges

$x = -1$: $-\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges

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So, $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $-1 \leq x \leq 1$

And we can then say that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \tan^{-1}(1) = \frac{\pi}{4}$$