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Series

Given a sequence $\{a_n\}$ we can create a new sequence

$$S_n = \sum_{i=1}^{n} a_i$$

Then we can ask what the limit of this sequence is. If the limit exists then we write

$$\sum_{i=1}^{\infty} a_i := \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} S_n$$

and we call this an <u>infinite series</u> or just a <u>series</u>

Further we call $S_n = \sum_{i=1}^{n} a_i$ the <u>n^{th} partial sum</u> of the series

<u>First example: Geometric Sequence and series</u>

We call the sequence $\{a_n\} = \{a r^{n-1}\}$ a <u>geometric sequence</u>, $a \neq 0$

$a_1 = a$, $a_2 = a \cdot r$, $a_3 = a \cdot r^2$, $a_4 = a \cdot r^3$, ...

or $\frac{a_{n+1}}{a_n} = r$ $a_1 = a$ (r is called the common ratio)

Then we call the series $\sum_{n=1}^{\infty} a r^{n-1}$ a <u>geometric series</u>

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What is this series? Look at the n^{th} partial sums:

$$S_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \dots + ar^{n-1}$$

In particular, look at

$$\begin{aligned}(1-r)S_n &= (1-r)(a + ar + ar^2 + \dots + ar^{n-1}) \\ &= a + ar + ar^2 + \dots + ar^{n-1} - ar - ar^2 - \dots - ar^{n-1} - ar^n \\ &= a - ar^n = a(1-r^n)\end{aligned}$$

So $S_n = \dfrac{a(1-r^n)}{1-r}$

UNLESS $r = 1$, then $S_n = n \cdot a$

So, we see that

$$\sum_{i=1}^{\infty} ar^{i-1} = \lim_{n\to\infty} \sum_{i=1}^{n} ar^{i-1} = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } r \geq 1 \text{ and } a > 0 \\ -\infty & \text{if } r \geq 1 \text{ and } a < 0 \\ \text{diverges} & \text{if } r \leq -1 \end{cases}$$

In particular, we can use this to represent the function $1/1-x$. That is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } -1 < x < 1$$

Theorems about series

① If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$
Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ or DNE then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof: $a_n = S_n - S_{n-1}$. So $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0$

② If $\{a_n\}$ is ultimately positive, then the series $\sum_{n=1}^{\infty} a_n$ either converges or diverges to infinity.

Proof: S_n is ultimately increasing, so if bounded converges and if not diverges to infinity.

③ a) $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$

b) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

c) if $a_n \leq b_n$ then $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$

Provided all limits actually exist

Convergence test for Positive Series

We have many tests that can help us determine when a series convergences (Remark: all the sequences will be positive so all the series will be increasing so it is enough to show bounded to ensure convergence)

1) The Integral test.

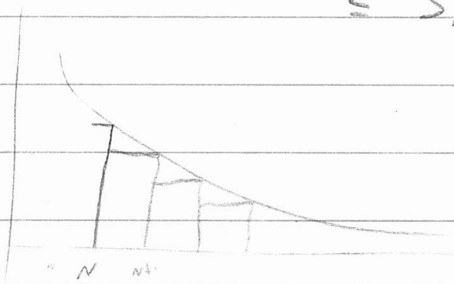
Suppose $a_n = f(n)$ When f is positive, continuous and nonincreasing on some interval $[N, \infty)$. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(t)\, dt$$

either both converge or diverge.

Proof: Choose $n \geq N$, then

$$S_n = a_1 + a_2 + \dots + a_N + a_{N+1} + a_{N+2} + \dots a_n$$
$$= S_N + f(a_{N+1}) + f(a_{N+2}) + \dots + f(a_n)$$
$$\leq S_N + \int_N^{\infty} f(t)\, dt$$



So if $\int_N^{\infty} f(t)\, dt$ converges ~~then~~ so does $\sum_{n=1}^{\infty} a_n$

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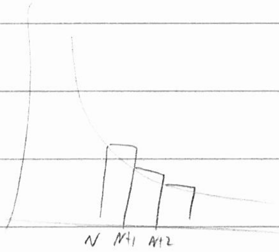
Conversely, we could have taken the left endpoints to get



$$\int_N^\infty f(t)\,dt \le f(N) + f(N+1) + \cdots +$$
$$= a_N + a_{N+1} + \cdots +$$
$$= S - S_{N-1}$$

And so if $\sum_{n=1}^{\infty} a_n$ converges (i.e. S exists) then $\int_N^\infty f(t)\,dt$ also converges

Example p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{array}$$

This is due to the fact that we can show that $\sum_{n=1}^{\infty} \frac{1}{n^p} \le S_N + \int_N^\infty \frac{1}{x^p}\,dx \underset{\text{Set } N=1}{=} 1 + \int_1^\infty \frac{1}{x^p}\,dx$

Caution: This tells us nothing about the true value of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ even though we can actually compute $\int_1^\infty \frac{1}{x^p}\,dx$. This only tells us it exists!

However, we can use this integral test to approximate our series.

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That is we have that if our series converges then have

$$s - s_N = \sum_{n=N+1}^{\infty} f(k) \leq \int_N^{\infty} f(x)\, dx$$

So if we can evaluate $\int_N^{\infty} f(x)\, dx$ then we can choose N large enough so that our error is small then compute S_n to get an approximation.

<u>Ex:</u> Approximate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to three decimal place

$$\sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{N} \frac{1}{n^4} \leq \int_N^{\infty} \frac{1}{x^4}\, dx = \frac{1}{3N^3}$$

So choose $N = 10$ and calculate that $\sum_{n=1}^{10} \frac{1}{n^4} = 1.08203658\ldots$

So $\sum_{n=1}^{\infty} \frac{1}{n^4} \approx 1.08203658$ up to three decimal places.

In fact, using more advanced technique one can show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.08232323\ldots$$

<u>Comment</u>: This shows that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞. This is a very important series and has a name: The Harmonic Series.

② The comparison test

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $0 \le a_n \le Cb_n$, ultimately, then

a) if $\sum_{n=1}^{\infty} b_n$ converges then so does $\sum_{n=1}^{\infty} a_n$

b) if $\sum_{n=1}^{\infty} a_n$ diverges then so does $\sum_{n=1}^{\infty} b_n$

(Note: this is slightly different than the Integral test as we could hat $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty} b_n$ diverges)

Proof: Straight forward.

We commonly compar series to geometric series or p-series

Ex: $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges since $\frac{1}{2^n+1} \le \frac{1}{2^n}$

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges since $\frac{1}{\ln n} \ge \frac{1}{n}$

③ Limit Comparison Test.

Suppose that $\{a_n\}$ and $\{b_n\}$ are positive sequences and that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ (L can be ∞)

a) If $L \neq \infty$ then if $\sum_{n=1}^{\infty} b_n$ converges then so does $\sum_{n=1}^{\infty} a_n$

b) if $L \neq 0$ then if $\sum_{n=1}^{\infty} b_n$ diverges then so does $\sum_{n=1}^{\infty} a_n$

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Comment: Makes intuitive sense since if $L \neq 0, \infty$ then a_n & b_n should be roughly the same size. While if $L = \infty$ then a_n is much bigger than b_n and so we can't say anything about a_n based off of b_n. Likewise for $L = 0$.

Ex: $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ "looks like $\frac{1}{\sqrt{n}}$" so guess diverges

$$\lim_{n\to\infty} \frac{\frac{1}{1+\sqrt{n}}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1$$

So by LCT it diverges since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-test

$\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ "looks like $\frac{1}{n^2}$" so guess converges.

Indeed $\lim_{n\to\infty} \frac{\frac{n+5}{n^3-2n+3}}{1/n^2} = 1$ so converges.

(4) The Ratio Test

Suppose that $a_n > 0$ ultimately and $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists or is ∞. Then

(a) If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges

(b) If $1 < \rho \leq \infty$ then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges

(c) If $\rho = 1$ then we can conclude nothing.

Comment This makes sense as if $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, then a_n "behaves" like ρ^n and so will converge or diverge based on the same principle of the geometric series.

This test is useful when we have exponents or factorials in our series

Ex: $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges since $\lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1$

(5) The Root Test

Suppose $a_n > 0$ ultimately and $\sigma = \lim_{n\to\infty} a_n^{1/n}$ exists or is $+\infty$.

a) $0 \le \sigma < 1$, then $\sum_{n=1}^{\infty} a_n$ converges

b) $1 < \sigma \le \infty$ then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverge to ∞.

c) $\sigma = 1$, this test says nothing

Comment Again makes sense because this implies than a_n "behaves" like σ^n and so follows the same rules as the geometric sequence