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Sequences and Series

Definitions A *sequence* is an ordered list of numbers indexed by the natural numbers.

Commonly we write it as in set notation

Ex: $\{1, 2, 3, 4, 5, \ldots\}$ or $\{2, 5, 4, 10, \pi, 1000000, \ldots\}$

Typically, when writting an arbitrary sequence we write $\{a_1, a_2, a_3, \ldots\}$. If there is a function f such that $f(n) = a_n$ for all natural numbers then we may just write $\{f(n)\}$

Ex: $\{n\} = \{1, 2, 3, 4, 5, \ldots\}$

$\{(1+\frac{1}{n})^n\} = \{2, (\frac{3}{2})^2, (\frac{4}{3})^3, (\frac{5}{4})^4, \ldots\}$

We can also define sequences recursively. That is, we can define what happens next by what happened before

Ex: $a_1 = 1$, $a_{n+1} = \sqrt{6+a_n}$

so $\{a_n\} = \{1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \ldots\}$

OR: $a_1 = 1$, $a_2 = 1$ $a_{n+1} = a_n + a_{n-1}$

$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ — Fibonacci Sequence.

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<u>Terms</u>: We say a sequence is <u>bounded below by L</u> if $a_n \geq L$ for all n and L is called a <u>lower bound</u>. We say a sequence is <u>bounded above by M</u> if $a_n \leq M$ for all n and M is called an <u>upper bound</u>. We say a sequence is <u>bounded by K</u> if $|a_n| \leq K$.

We say the sequence is <u>positive</u> if it is bounded below by 0 and <u>negative</u> if it is bounded above by 0.

We say the sequence is <u>increasing</u> if $a_{n+1} \geq a_n \; \forall n$ and <u>decreasing</u> if $a_{n+1} \leq a_n$ for all n and <u>monotonic</u> if it is either increasing or decreasing.

We say that a sequence is <u>ultimately increasing</u> or <u>ultimately decreasing</u> if after a finite amount of time it is increasing or decreasing respectively. (note: increasing / decreasing sequences are also ultimately increasing / decreasing)

Ex: $\left\{\frac{n^2}{2^n}\right\} = \left\{\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \frac{36}{64}, \ldots\right\}$ then we have $a_1 < a_2 < a_3$ but $a_3 > a_4 > a_5 > \ldots$

So while it is not always decreasing, eventually it is.

We say a sequence is <u>alternating</u> if $a_n a_{n+1} < 0$. That is, if the sign always changes.

Definition Limit of a Sequence.

We say the sequence $\{a_n\}$ converges to the limit L and write $\lim_{n\to\infty} a_n = L$ if for all $\varepsilon > 0$ there exists an N such that for all $n \geq N$, $|a_n - L| < \varepsilon$.

Informally, this just means that as n gets bigger, a_n gets closer to L.

If the limit does not exist, we say it diverges or diverges to ∞ or $-\infty$

Ex:

$\{a_n\} = \left\{\frac{n-1}{n}\right\}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{n} = 1$

$\{a_n\} = \{n^2\}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$

$\{a_n\} = \{(-1)^n\}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n$ diverges

$\{a_n\} = \{(-1)^n n^2\}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n n^2$ diverges (but not to ∞ or $-\infty$)

As one may have guessed if there is a function $f(x)$ such that $a_n = f(n)$ and if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Therefore for $\{a_n\} = \left\{\left(1+\frac{1}{n}\right)^n\right\}$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Properties of these limits

1. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\lim_{n\to\infty} a_n\right) / \left(\lim_{n\to\infty} b_n\right)$ Provided $\lim_{n\to\infty} b_n \neq 0$
4. If $a_n \leq b_n$ ultimately, then
$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$
5. If $a_n \leq b_n \leq c_n$ ultimately and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$
Then $\lim_{n\to\infty} b_n = L$ (squeeze theorem).

__Thm__ If $\{a_n\}$ converges, then it is bounded.

__Proof__ If $\lim_{n\to\infty} a_n = L$. then by definition there exist an N such that if $n \geq N$ then $|a_n - L| \leq 1$ and so $|a_n| \leq 1+L$ for $n \geq N$.
Moreover if we set $K = \max_{n \leq N} |a_n|$ (which is finite) then we get $|a_n| \leq \max(K, 1+L)$.

__Thm__ If the sequence $\{a_n\}$ is bounded above and is ultimately increasing then it converges.
Likewise, if it is bounded below and ultimately decreasing then it also converges.

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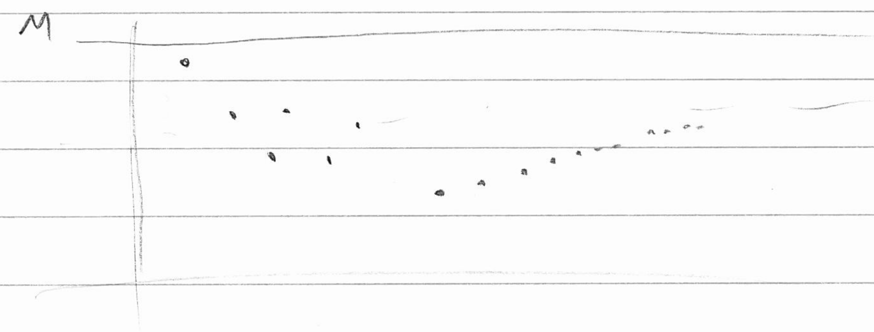
"proof by picture"



Ex: Show that the sequence $a_1 = 1$ $a_{n+1} = \sqrt{6 + a_n}$ converges.

We will show it is bounded above and increasing and so, by the theorem it converges.

To show increasing we use induction:

1. $a_2 = \sqrt{7} \geq 1 = a_1$
2. Assume $a_{n+1} \geq a_n$
3. $a_{n+2} = \sqrt{6 + a_{n+1}} > \sqrt{6 + a_n} = a_{n+1}$

So increasing.

Now to show bounded above: Again by induction

1. $a_1 = 1 \leq 3$
2. Assume $a_n \leq 3$
3. $a_{n+1} = \sqrt{6 + a_n} \leq \sqrt{6 + 3} = 3$

So bounded by 3.

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We can then use the fact that we know it converges to find the limit. That is, we have

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6+a_n} = \sqrt{6+a}.$$

So $0 = a^2 - a - 6 = (a-3)(a+2) \Rightarrow a = 3$ or -2

but obviously $\underline{a=3}$

so $\lim_{n\to\infty} a_{n+1} = 3.$

Two important limits:

① If $|a|<1$, then $\lim_{n\to\infty} a^n = 0$

We already saw that $\lim_{\cancel{x}\to\infty} a^x = 0$ if $|a|<1$.

② If a is any real number then $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Pick $N > |x|$, then if $n \geq N$, we have

$$\left|\frac{x^n}{n!}\right| = \frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\cdots\frac{|x|}{N-1}\frac{|x|}{N}\frac{|x|}{N+1}\cdots\frac{|x|}{n}$$

$$\leq \frac{|x|^{N-1}}{(N-1)!}\frac{|x|}{N}\cdot\frac{|x|}{N}\cdots\frac{|x|}{N} = \frac{|x|^{N-1}}{(N-1)!}\cdot\left(\frac{|x|}{N}\right)^{n-N+1}$$

$$= \frac{N^{N-1}}{(N-1)!}\left(\frac{|x|}{N}\right)^n \longrightarrow 0$$

Since $|x| < N$

and so $\frac{|x|}{N} < 1$