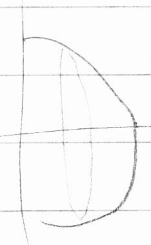
Ex: Find the surface area of the sphere of radius r.

As always it is enough to find the surface area of the semisphere obtained by rotating the quarter circle



$\leftarrow y = \sqrt{r^2 - x^2} = f(x)$ $\qquad f'(x)^2 = \frac{x^2}{r^2 - x^2}$

$$S = \int_0^r 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= 2\pi r \int_0^r dx = 2\pi r^2$$

So surface area of a sphere is $4\pi r^2$.

Parametric Curves

Sometimes it can be difficult to describe how the y-values of a graph depend on the x-values of the graph. It can sometimes be useful to just write how the x and y-values are changing with respect to a different parameter (i.e. time).

Definition A parametric curve $\mathcal{C}$ in the plane consists of an ordered pair (f, g) of continuous functions both defined on the same interval I

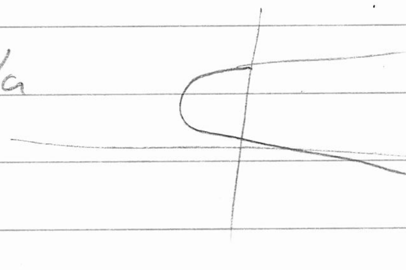
The graph of this curve can be obtained by drawing all the points (x,y) satisfying
$x = f(t)$, $y = g(t)$ for $t \in I$.

These are called the <u>parametric equations</u> of $\mathcal{C}$ and the independent variable t is called the <u>parameter</u>. The graph is called the <u>plane curve</u> and any pair of functions (f, g) that generate the same graph is called a <u>parametrization</u>

Ex: Sketch the curve given by $x = t^2 - 1$, $y = t + 1$

We can rearrange to find $t = y - 1$ and so get that
$x = (y-1)^2 - 1 = y^2 - 2y + 1 - 1 = y^2 - 2y$

And so it is a sideways parabola



Since we tend to consider t as a time parameter, we should think of the curve as "moving" from a smaller time to a larger one. Therefore, the parametric curve has a "direction" and we should indicate this with arrows

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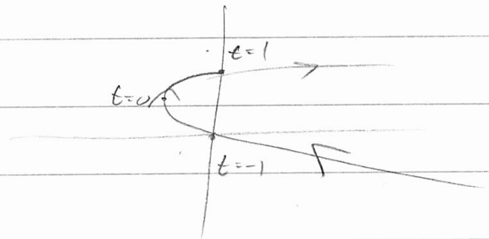
So, the curve $x = t^2 - 1$, $y = t + 1$ should be drawn as

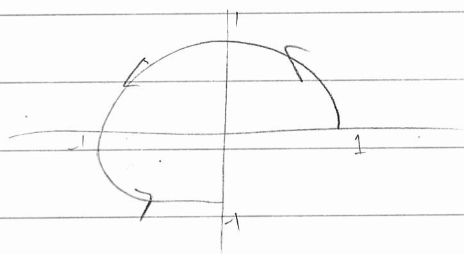


In general if $x = f(t)$ and $y = g(t)$ then we can write $y = g(f^{-1}(x))$, if f is one-to-one.

Ex: Plot the graph $x(t) = \cos(t)$, $y(t) = \sin(t)$
$0 \le t \le 3\pi/2$

If $0 \le t \le \pi$ then $t = \cos^{-1}(x)$ and
so $y(t) = \sin(\cos^{-1}(x)) = \sqrt{1-x^2}$

However, this doesn't tell us what happens between π & $\frac{3\pi}{2}$. For this we need the observation that $\cos^2 t + \sin^2 t = 1$. That is, the graph is the graph of $x^2 + y^2 = 1$.
But not the whole circle. At $t=0$: $(x,y) = (1,0)$
At $t = \frac{3\pi}{2}$, $(x,y) = (0,-1)$ so the graph looks like



(119)

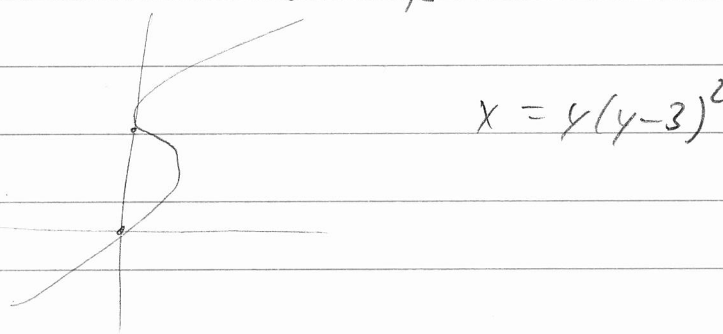
Since f^{-1} may not always exist, it is useful to find a relationship between x & y

Ex: Sketch $x = t^3 - 3t$, $y = t^2$, $-2 \leq t \leq 2$

Note that $x = t(t^2-3) = t \cdot (y-2)$

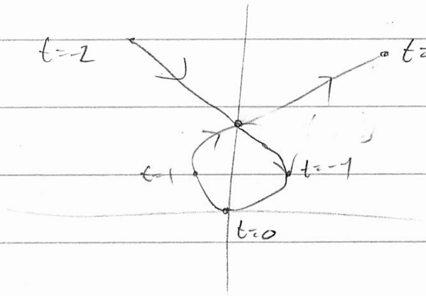
Therefore $x^2 = t^2(y-3)^2 = y(y-3)^2$

To sketch this first consider $x = y(y-3)^2$.
This will look like a sideways cubic with roots at 0 & 3.
ie.



Now, for point with x positive there are two possible points on $x^2 = y(y-3)^2$ and for every point with x negative then on no points on $x^2 = y(y-3)^2$

So:



Then we check in what direction it is moving

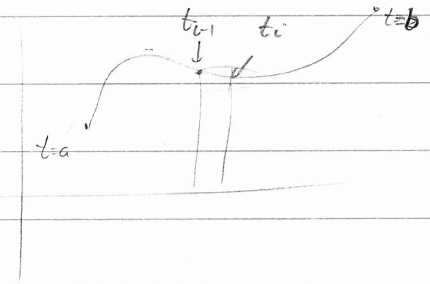
t	-2	-1	0	1	2
x	-2	2	0	-2	2
y	4	1	0	1	4

(120)

Area bounded by parametric curves

Consider a parametric curve with graph

$x = f(t) \quad y = g(t)$



What is the area under it?

We split the interval $[a,b]$ into subintervals of t

So area of our estimating rectangles are

$g(t_i)\,(f(t_i) - f(t_{i-1}))$ and so

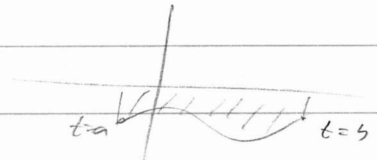
$$A \approx \sum_{i=1}^{n} g(t_i) \frac{(f(t_i) - f(t_{i-1}))}{t_i - t_{i-1}} \Delta t_i$$

$$\text{So } A = \int_a^b g(t)\, f'(t)\, dt = \int_a^b y\, dx$$

This picture is a little misleading as we are always above the x-axis ($g > 0$) and our x-values are always increasing ($f' > 0$)

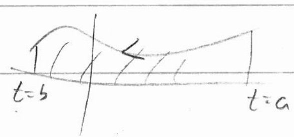
So, we get three cases:

(1) $f'(t) > 0, \quad g(t) < 0$



$$A = -\int_a^b g(t) f'(t)\, dt$$

(2) $f'(t) < 0, \quad g(t) > 0$



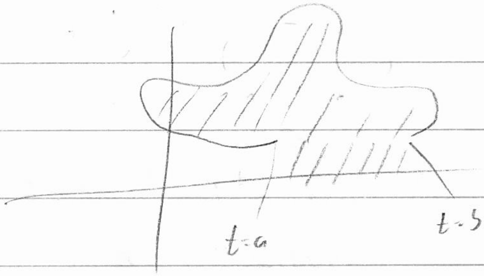
$$A = -\int_a^b g(t) f'(t)\, dt$$

(3) $f'(t) < 0, \quad g(t) < 0$

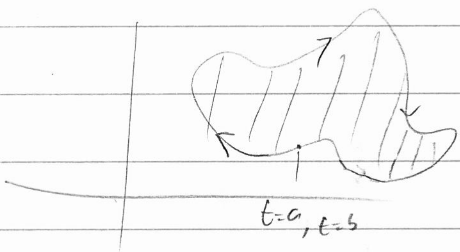
$$A = \int_a^b g(t)\, f'(t)\, dt.$$

So, really $\int_a^b g(t)\, f'(t)\, dt$ is calculating the area **bounded** by the parametric curve and the x-axis.

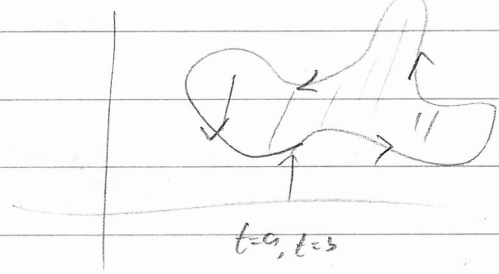
Ex:



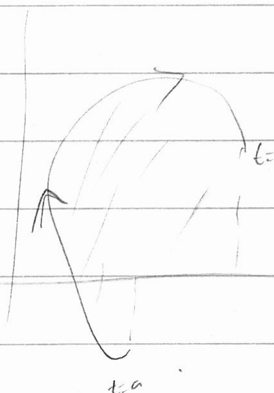
$$\int_a^b g(t)\, f'(t)\, dt = A$$



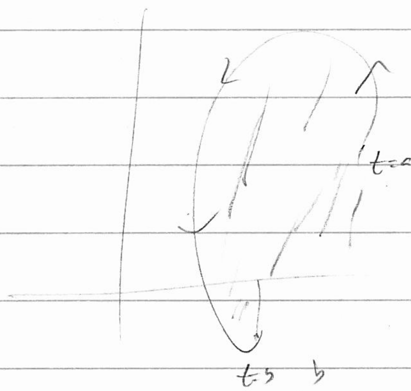
$$\int_a^b g(t)\, f'(t)\, dt = A$$



$$-\int_a^b g(t)\, f'(t)\, dt = A$$



$$A = \int_a^b g(t)\, f'(t)\, dt$$



$$A = -\int_a^b g(t)\, f'(t)\, dt$$

In conclusion:

$$A = \int_a^b g(t)\, f'(t)\, dt \quad \text{if } \mathcal{C} \text{ is transversed clockwise}$$

while,

$$A = -\int_a^b g(t)\, f'(t)\, dt \quad \text{if } \mathcal{C} \text{ is transversed counterclockise.}$$

Exercise: Show that $\int_a^b x\, dy$ "

$\int_a^b f(t)\, g'(t)\, dt$ is the area

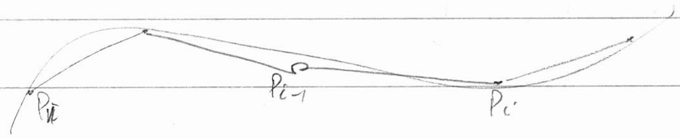
bounded by the curve ang the Y-axis

in the same way.

(12.3)

Arclength of parametric curves

If we do the same process for parametric curve to find the arclength we get



$$S \approx \sum_{i=1}^{n} |P_{i-1} P_i|$$

But here our points are not in the form $(x_i, f(x_i))$ but in the form $(f(t_i), g(t_i))$ where our parametric curve is given by $x = f(t)$, $y = g(t)$.

Therefore,

$$|P_{i-1}P_i| = \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}$$

and

$$S \approx \sum_{i=1}^{n} \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right)^2 + \left(\frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}\right)^2} \Delta t$$

And hence:

$$S = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} \, dt.$$

Since $x = f(t)$ and $y = g(t)$ $f'(t) = \frac{dx}{dt}$ and $g'(t) = \frac{dy}{dt}$

So we commonly just write

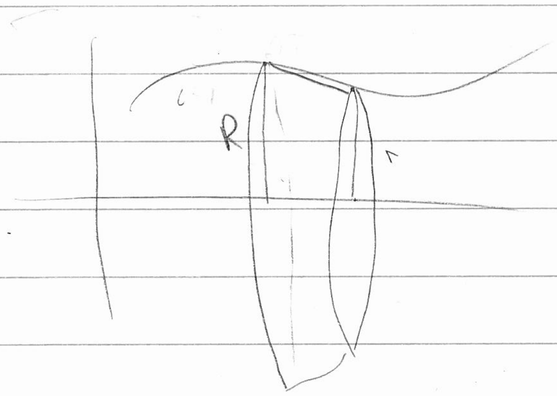
$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

Therefore by the FTC, we see that

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Surface area

Again if we rotate about the x-axis, we can calculate surface area:



$x = f(t), \quad y = g(t)$

$$S \approx \sum_{i=1}^{n} \pi\left(|g(t_i)| + |g(t_{i-1})|\right) \sqrt{\left(\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right)^2 + \left(\frac{g(t_i) + g(t_{i-1})}{t_i - t_{i-1}}\right)^2}\ \Delta t$$

$$S = \int 2\pi |g(t)| \sqrt{f'(t)^2 + g'(t)^2}\ dt$$

We can simplify this by noting that

$$ds = \sqrt{f'(t)^2 + g'(t)^2}\, dt \quad \text{and} \quad y = g(t) \quad \text{and so}$$

write that the surface area obtained by rotating about the x-axis is

$$S = 2\pi \int_a^b |y| \, ds$$

In a similar way, we can write the surface area obtained by rotating about the y-axis as

$$S = 2\pi \int_a^b |x| \, ds$$

Exercise: Show this.