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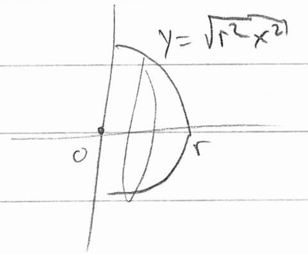
Example Find the volume of a sphere of radius r.

First need only find radius of semisphere and then double.
For semisphere, need only rotate a quarter circle.



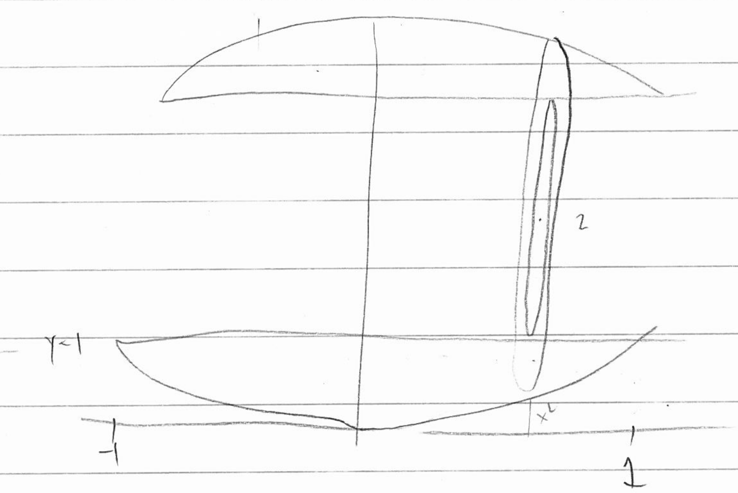
$$V = 2\int_0^r \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$

$$= 2\pi\int_0^r r^2-x^2\,dx = 2\pi\; r^2x-\frac{x^3}{3}\Big|_0^r$$

$$= 2\pi\left(r^3-\frac{r^3}{3}\right) = \frac{4}{3}\pi r^3.$$

We also need not rotate about an axis.
Nor do we need the line to be attach to the object we are rotating

Ex: Find the volume of the ring generated by rotating the region bounded by $y=x^2$ and the line $y=1$ about the line $y=2$.



What is the area of a single shape?
It is the area of the larger circle minus the area of the smaller circle.

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Area of larger circle: $\pi(2-x^2)^2$

Area of smaller circle: $\pi 1^2$

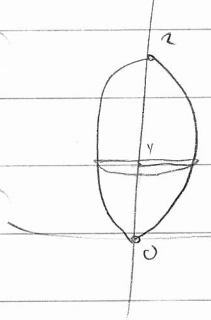
Area of one shape: $\pi((2-x^2)^2-1) = \pi(3-4x^2+x^4)$

So, volume of entire shape is

$$V = \int_{-1}^{1} \pi(3-4x^2+x^4)\,dx = \frac{56\pi}{15}.$$

We can also integrate with respect to the y-variable

Ex: Find the volume generated by rotating the region bounded by the curve $x = 2y-y^2$ and the y-axis

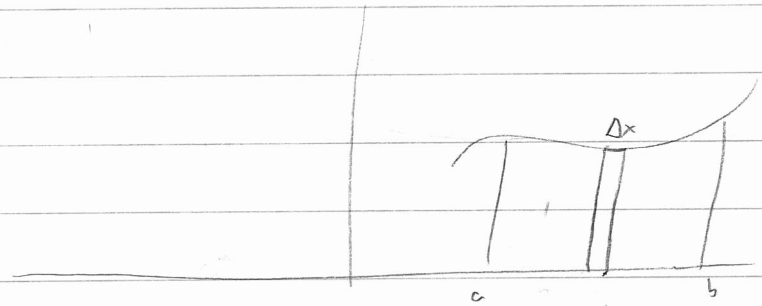


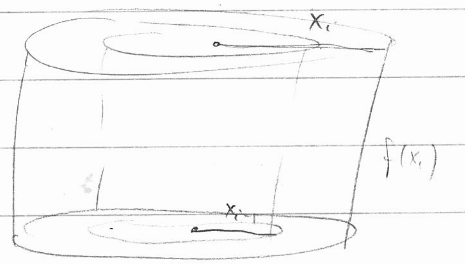
$$r(y) = 2y-y^2$$

$$V = \int_0^2 \pi(2y-y^2)^2\,dy = \frac{16\pi}{15}.$$

Cylindrical Shells

Suppose we took the region under a curve and rotated it about the y-axis. What would an estimating rectangle look like?





Volume of outer cylinder $= \pi x_i^2 f(x_i)$

Volume of inner cylinder $= \pi x_{i-1}^2 f(x_i)$

Volume of shape: $\pi f(x_i) \quad x_i^2 - x_{i-1}^2$

$= \pi f(x_i) \Delta x_i^2$

So $V \approx \sum_{i=1}^{n} \pi f(x_i) \Delta x_i^2$. Taking $\Delta x_i \to dx$ get

$$V = \int_a^b \pi f(x) \, dx^2$$ But $dx^2 = 2x \, dx$

So volume of cylindrical shell is

$$V = 2\pi \int_a^b x f(x) \, dx$$

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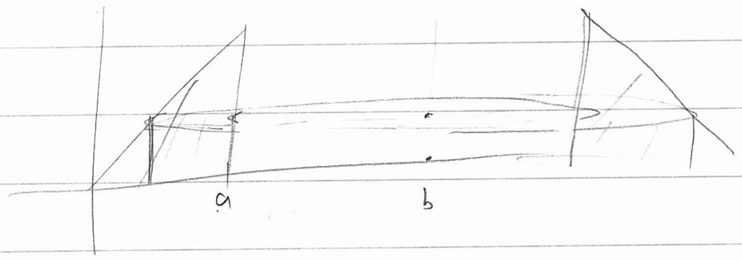
Ex: Find the volume of the region bounded by $y=x$, $y=0$, $x=a$ and rotated around the line $x=b>a$

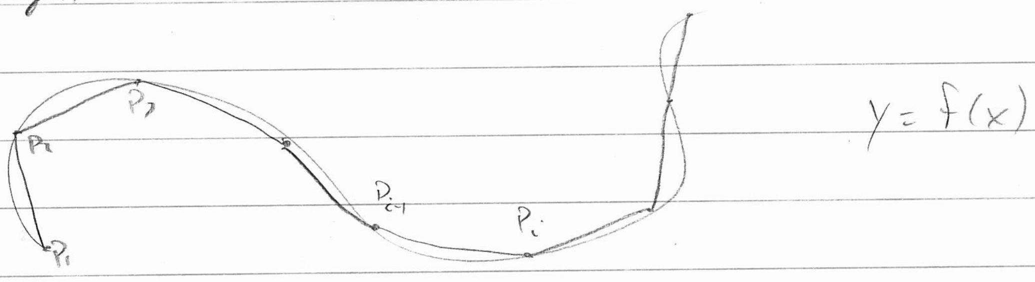


Cylindrical shell at radius $b-x$ and height x as x goes from 0 to a.

Therefore $V = 2\pi \int_0^a (b-x)\,x\,dx = \pi\left(a^2 b - \frac{2a^3}{3}\right)$

Arc length

Given a curve, we want to determine its length. We can use the same idea as integration. That is we approximate it with straight lines



Denote $|P_{i-1}P_i|$ as the length of the line joining P_{i-1} & P_i

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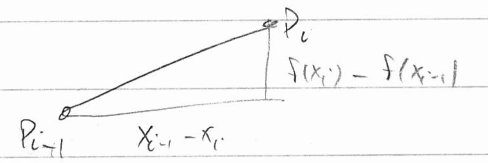
Then we get $S \approx \sum_{i=1}^{n} |P_{i-1} P_i|$

Note that we can say $P_i = (x_i, f(x_i))$.
And so:



and

$$|P_{i-1} P_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \; \Delta x_i$$

So $$S \approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \; \Delta x_i$$

As we take $n \to \infty$ the $\Delta x_i \to dx$ but what happens to $\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$?

It will tend to $f'(x)$, Hence:

$$S = \int_a^b \sqrt{1 + (f'(x))^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Ex: Find the length of the curve $y = x^{2/3}$ from $x = 1$ to $x = 8$

$\frac{dy}{dx} = \frac{2}{3} x^{-1/3}$ so

$$S = \int_1^8 \sqrt{1 + \frac{4}{9} x^{-2/3}}\, dx = \int_1^8 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx$$

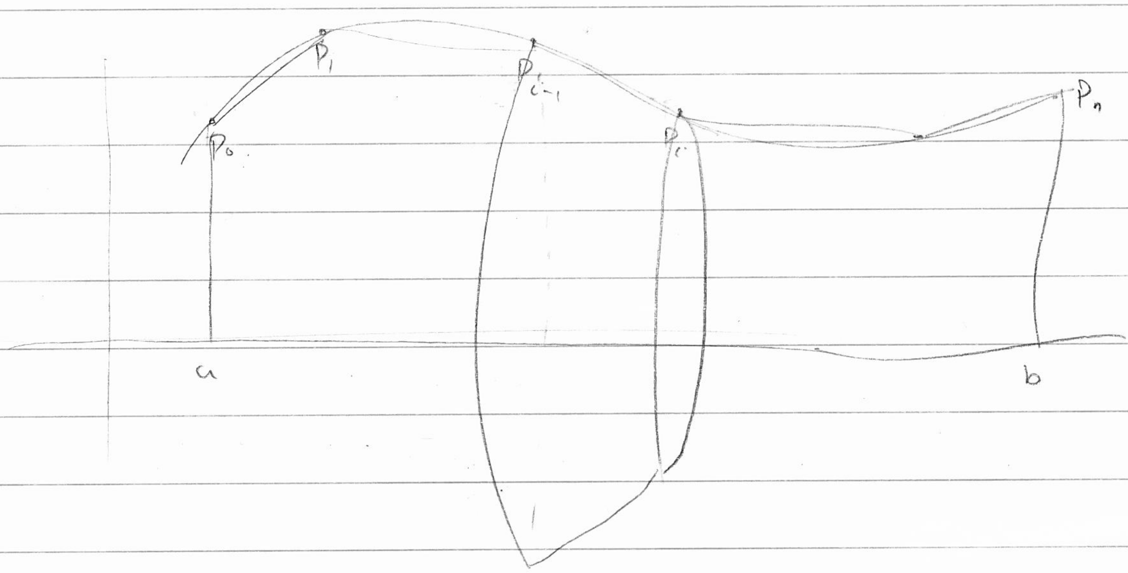
$u = 9x^{2/3} + 4$
$du = 6x^{-1/3}\, dx$

$$= \int_{13}^{40} \frac{\sqrt{u}}{3} \frac{du}{6} = \frac{1}{27} u^{3/2} \Big|_{13}^{40}$$

$$= \frac{40\sqrt{40} - 13\sqrt{13}}{27} \approx 7.6337\ldots$$

Surface area

We can use the same ideas for determining the arclength to determine the surface area of an object rotated about an axis



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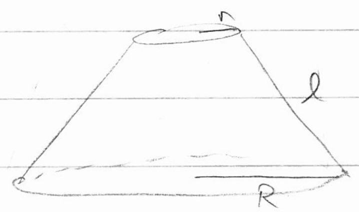
We see that the shapes we get when rotating our lines that approximate our curves are frustrum or a truncated cone. That is it has the shape



Exercise: Show the surface area of such a frustrum is $\pi(R+r)l$.

Therefore we see that our surface area will be approximated by.

$$S \approx \sum_{i=1}^{n} \pi\left(f(x_{i-1}) + f(x_i)\right) \left| P_{i-1} P_i \right|$$

$$= \sum_{i=1}^{n} \pi\left(f(x_{i-1}) + f(x_i)\right) \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \, \Delta x_i$$

Taking the limit $n \to \infty$ we get

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx$$