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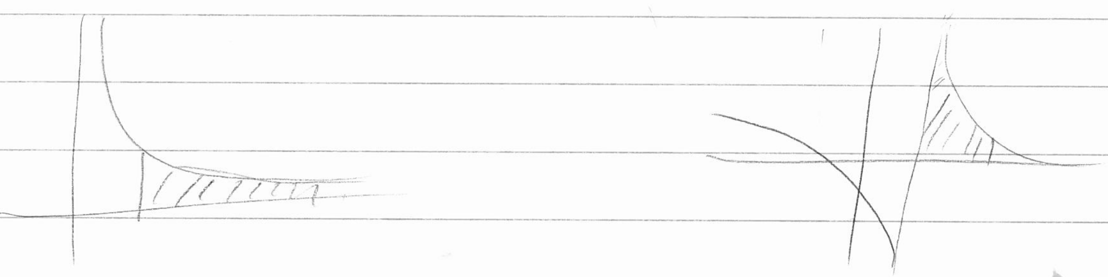
Improper Integrals

Up until now we have only considered what happens when our functions and delimiters are bounded.
Can the integral make sense even if it is unbounded?

i.e.



Can either of these integrals exist? Be finite?

Type I improper integrals: Infinite delimiters

We define $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

and $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

In either case, if the limit exists, we say the improper integral converges. If not it diverges or diverges to ∞ or $-\infty$ if the limit is ∞ or $-\infty$

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Ex: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(\frac{-1}{x}\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{-1}{b} - \left(\frac{-1}{1}\right)\right)$

$= 1$

But $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\Big|_1^b = \lim_{b\to\infty} \ln(b) = \infty.$

In general: If $0 < a < \infty$, then

$$\int_a^\infty \frac{1}{x^p}\,dx \begin{cases} \text{converges to } a^{1-p}/p-1 & \text{if } p > 1 \\ \text{diverges to } \infty & \text{if } p \leq 1 \end{cases}$$

These are called p-integrals

[Makes intuitive sense that should diverge for $p < 0$ as $\frac{1}{x^p}$ tends to ∞

Ex: $\int_0^\infty \cos x\,dx = \lim_{b\to\infty} \sin b$ diverges

Type two improper integrals: Unbounded function.
If f is unbounded at a then we define

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

Likewise, if f is unbounded at b then we define

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

Similarly, we say this improper integral converges, diverges or diverges to ∞ or $-\infty$ depending on the limit.

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Ex: $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1 = \lim_{a\to 0^+} (2\sqrt{1} - 2\sqrt{a}) = 2.$

In general: if $0 < a < \infty$, then

$$\int_0^a \frac{1}{x^p}dx \begin{cases} \text{converges to } a^{1-p}/1-p & \text{if } p<1 \\ \text{diverges to } \infty & \text{if } p \geq 1 \end{cases}$$

These are also callede p-integrals

Note that the ones that converge are the opposite of those that converge for the type I integrals

Also note that $1/x$ diverges for both. This is because $\ln x$ tense to ∞ at ∞ and $-\infty$ at 0.

Ex: $\int_0^1 \ln x \, dx = \lim_{a\to 0^+} \int_a^1 \ln(x)\,dx = \lim_{a\to 0^+} x\ln x - x\Big|_a^1$

$= \lim_{a\to 0^+} 1 \cdot \ln 1 - 1 - a\ln a - a = -1 - \underset{\text{LHR}}{\lim_{a\to 0^+} a\ln a} = -1$

Sometimes it is not easy to actually compute the improper integral but we can still show that it converges.

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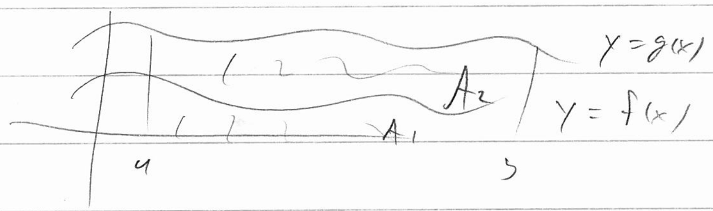
Comparison Theorem for Integrals

Let $-\infty \leq a < b \leq \infty$, and suppose that $0 \leq f(x) \leq g(x)$.
If $\int_a^b g(x)\,dx$ converges then $\int_a^b f(x)\,dx$ converges

and $$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Also if $\int_a^b f(x)\,dx$ diverges then so does $\int_a^b g(x)\,dx$

'Proof':



$A_1 \leq A_2$

Ex $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{-1} \frac{1}{1+x^2}\,dx + \int_{-1}^{1} \frac{1}{1+x^2}\,dx + \int_{1}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\leq \int_{-\infty}^{-1} \frac{1}{x^2}\,dx + \int_{-1}^{1} \frac{1}{1+x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx$$ converges

Further $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \lim_{a\to-\infty} \int_a^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \lim_{a\to-\infty} \tan^{-1}(b) - \tan^{-1}(a)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Ex: $$\int_0^{\infty} e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx$$

$$\leq \int_0^1 1\,dx + \int_1^{\infty} e^{-x}\,dx$$

$$= 1 + \lim_{b\to\infty} \frac{-1}{e^b} - \frac{-1}{e} = 1 + \frac{1}{e}$$

Remark: $\int e^{-x^2}\,dx$ occurs often in probability and statistics. However, it does not have a more simple representation. That is, there is no simple antiderivative of e^{-x^2}. Thus we sometimes will write

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x) = \text{error function}$$

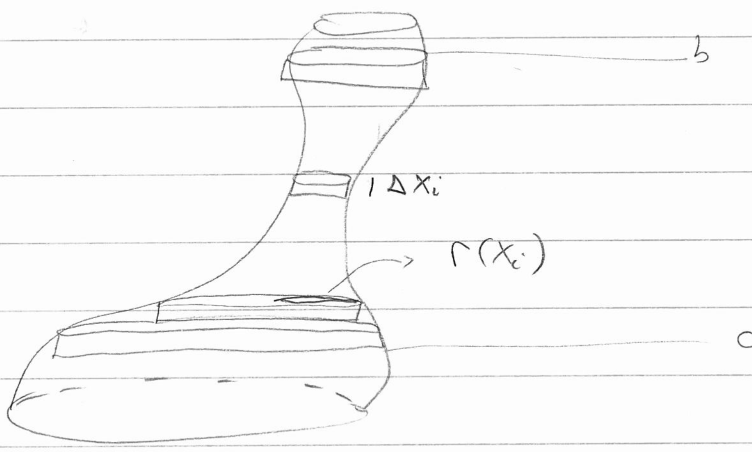
This $\frac{\sqrt{\pi}}{2}$ comes from the fact that $\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$. There is actually a very simple proof of this using multivariable calculus which we will not do.

Section 5: Applications

Volumes

We can calculate volumes of three dimensional shapes in much the same way we calculated areas under curves.

If we have a shape that we want to find the volume of we can approximate it by cylinders by slicing it



So, if we want to know the volume of this shape from height (x) a to height b and we know that the radius can be written as a function of the height as r(x) then at height x_i the cylinder will have volume

$$V(x_i) = A(x_i)\,\Delta x_i = \pi r(x_i)^2\,\Delta x_i$$

And so summing them all, we find that the volume will be approximately

$$V \approx \sum_{i=1}^{n} V(x_i) = \sum_{i=1}^{n} A(x_i)\,\Delta x_i = \sum_{i=1}^{n} \pi\, r(x_i)^2\,\Delta x_i$$

Taking the intervals arbitrarily small (or, equivalently, $n \to \infty$), we get

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\, r(x)^2\,dx$$

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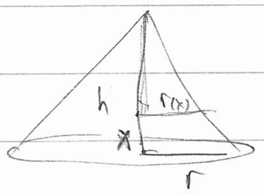
Ex: Volume of a cone of height h and base radius r.



$$\frac{r(x)}{h-x} = \frac{r}{h} \Rightarrow r(x) = \frac{r}{h}(h-x)$$

$$\Rightarrow V = \int_0^h \pi r(x)^2 \, dx = \pi \frac{r^2}{h^2} \int_0^h (h-x)^2 \, dx$$

$$= \pi \frac{r^2}{h^2} \int_0^h h^2 - 2xh + x^2 \, dx = \pi \frac{r^2}{h^2} \left[h^2 x - x^2 h + \frac{x^3}{3} \Big|_0^h \right]$$

$$= \pi \frac{r^2}{h^2} \left[h^3 - h^3 + \frac{h^3}{3} \right] = \frac{\pi r^2 h}{3}$$

We can also use this to find the volume of an object obtained by rotating a function around a line.

Then the radius function is f(x) and we get

$$V = \int_a^b \pi f(x)^2 \, dx.$$