95

Integrals of Rational Functions

We have already seen or can deduce from substitution method that

① $\int \frac{1}{ax+b} = \frac{1}{a} \ln |ax+b| + C$

② $\int \frac{x}{x^2+a^2} dx = \frac{1}{2} \ln (x^2+a^2) + C$

③ $\int \frac{x}{x^2-a^2} dx = \frac{1}{2} \ln |x^2-a^2| + C$

④ $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

But what about $\int \frac{1}{x^2-a^2} dx$?

We do this by partial fractions. That is since $x^2-a^2 = (x-a)(x+a)$ factors we can find an A & B such that

$$\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}$$

In particular $A = \frac{1}{2a}$, $B = -1/2a$.

(96)

So, $\int \frac{1}{x^2-a^2}\,dx = \int \frac{1/2a}{x-a} - \frac{1/2a}{x+a}\,dx$

$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$

$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

In general, if we have a rational function $\frac{P(x)}{Q(x)}$ where $Q(x)$ factors into linears then

$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ then

we can find $A_1, A_2, \ldots A_n$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_n}{(x-a_n)}$$

And hence $\int \frac{P(x)}{Q(x)}\,dx = A_1\ln|x-a_1| + \cdots + A_n\ln|x-a_n| + C.$

Ex: $\frac{x+4}{x^2-5x+6} = \frac{-6}{x-2} + \frac{7}{x-3} \Rightarrow \int \frac{x+4}{x^2-5x+6}\,dx = -6\ln|x-2| + 7\ln|x-3| + C$

What if $Q(x)$ does not factor into linears?

97

Ex: $\int \frac{x+2}{x^2+2x+3}\,dx = \int \frac{x+2}{(x+1)^2+2}\,dx = \int \frac{u+1}{u^2+2}\,du$

$= \int \frac{u}{u^2+2}\,du + \int \frac{1}{u^2+2}\,du = \frac{1}{2}\ln(u^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

Anothe issue: $Q(x)$ has repeated factors

Ex: $\int \frac{1}{x(x-1)^2} = \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$

$= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C$

Partial fraction decomposition theorem

Let P and Q are polynomials with the degree of P less than the degree of Q. Further suppose

$$Q(x) = (x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}$$

Then $\frac{P(x)}{Q(x)}$ can be ~~written~~ expressed as a sum of partial fractions of the form

$$\frac{A_1}{x-a_i} + \frac{A_2}{(x-a_i)^2} + \cdots + \frac{A_{m_i}}{(x-a_i)^{m_i}}$$

and

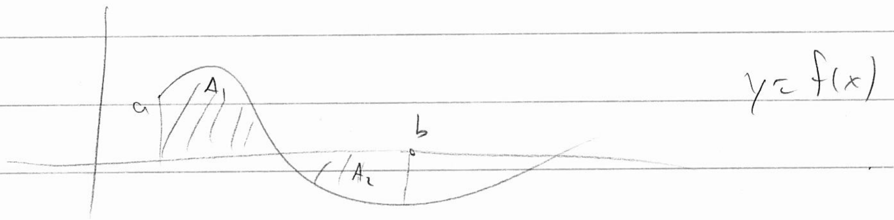
$$\frac{B_1x+C_1}{x^2+b_ix+c_i} + \cdots + \frac{B_{n_i}x+C_{n_i}}{(x^2+b_ix+c_i)^{n_i}}$$

To integrate P/Q, we integrate each term separately. To integrate the quadratic terms we must complete the squares.

Areas of Plane Regions

The integral is the area under the curve. However, if the curve becomes negative, then the approximating rectangle heights also become negative. That is our integral will be negative.

That is if our curve looks like



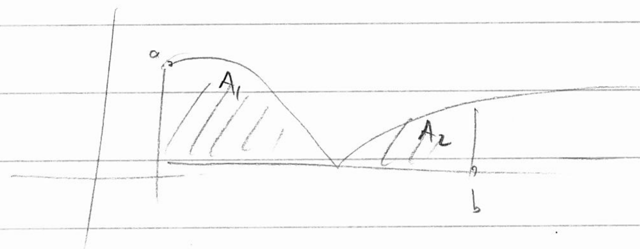
Then $\int_a^b f(x)\,dx = A_1 - A_2$.

So sometimes we will want to know the area bounded by the curve and the x-axis. That is, we want to know $A_1 + A_2$.

To do this we need to integrate $|f(x)|$.

(99)

That is $y = |f(x)|$ has the shape



and so $\displaystyle\int_a^b |f(x)|\,dx = A_1 + A_2$

Unfortunately there is no way straight forward way to calculate the integral of an absolute value. The only way is to split it into two integrals

That is $\displaystyle\int_a^b |f(x)|\,dx = \int_a^c |f(x)|\,dx + \int_c^b |f(x)|\,dx$

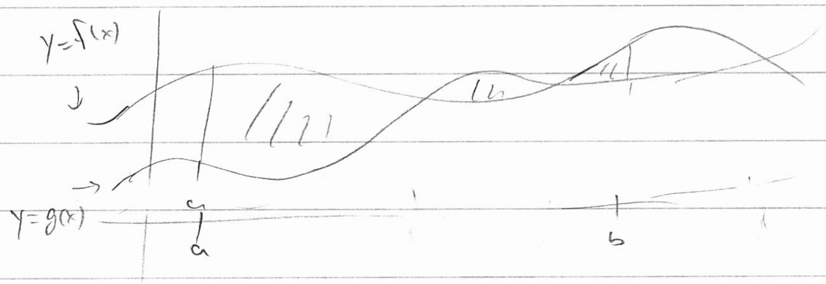
$$= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

Ex: Find the area bounded by $y = \cos x$ and the x-axis from 0 to $3\pi/2$.

$$\int_0^{3\pi/2} |\cos x|\,dx = \int_0^{\frac{\pi}{2}} |\cos x|\,dx + \int_{\frac{\pi}{2}}^{3\pi/2} (-\cos x)\,dx$$

$$= \sin x \Big|_0^{\frac{\pi}{2}} - \sin x \Big|_{\pi/2}^{3\pi/2} = (1-0) - (-1-1) = 3$$

It need not always be bounded by the x-axis
We could ask to find the area bounded between
two curves from a to b.

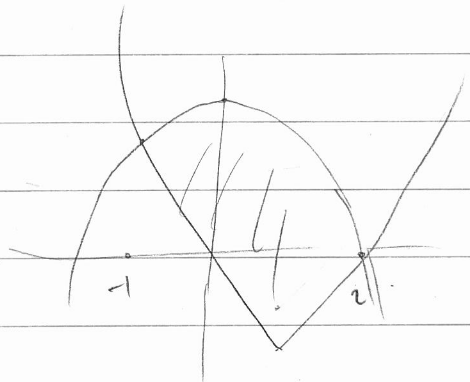


Notice if $f(x) > g(x)$ then the height at the rectangles approximating the area will be $f(x)-g(x)$
while if $f(x) < g(x)$ the height will be $g(x)-f(x)$.
In either case the height will be $|f(x)-g(x)|$
and we get

$$\int_a^b |f(x)-g(x)|\,dx$$

Moreover, the endpoints need not be expressed.

Ex: Find the area bounded by $y=x^2-2x$ & $y=4-x^2$



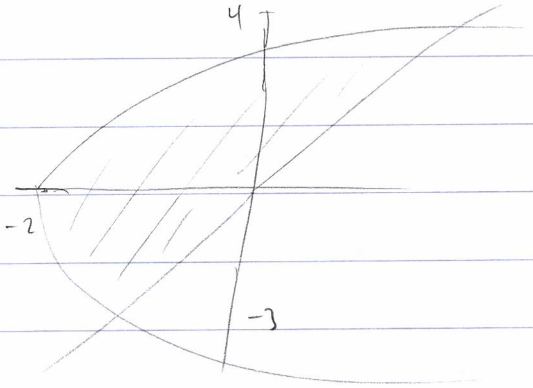
$$A = \int_{-1}^{2} 4-x^2-(x^2-2x)\,dx$$

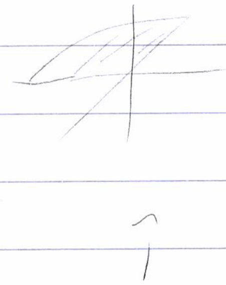
$$= \int_{-1}^{2} 4-2x^2+2x\,dx$$

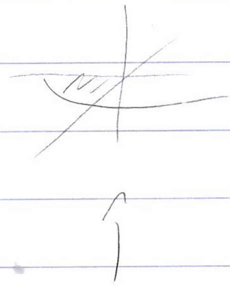
$$= 9$$

Finally, we may consider integrating along the y-axis to make the question more manageable.

Ex: Find the area bounded by $x=y^2-12$ & $y=x$.







We could split it into two case $y=\sqrt{x+12}$ & $y=-\sqrt{x+12}$
But an easier way would be to integrate with respect to y. That is:

$$\int_{-3}^{4} y-(y^2-12)\,dy = \frac{y^2}{2}-\frac{y^3}{3}+12y\Big|_{-3}^{4} = \frac{343}{6}$$