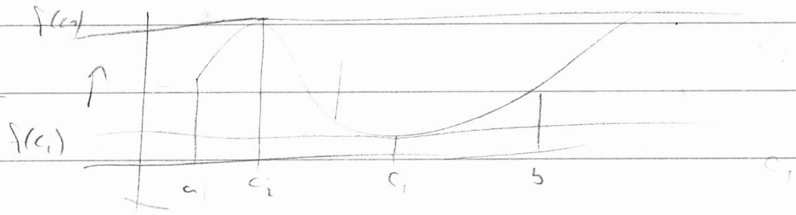
(87)

**Proof:**




$$f(c_1)\,(a-b) \le \int_a^b f(x)\,dx \le f(c_2)(a-b)$$

As we increase $c_1 \to c_2$ eventually get equality. ∎

Again, recall that $\ln(x)$ is defined as the area under $1/t$ from $1$ to $x$. So we may write

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

Moreover, we saw that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.

In fact this is true in general

Fundamental Theorem of Calculus:
Suppose $f$ is a continuous function on an interval $I$ containing $a$.

(1) Let $F(x) = \int_a^x f(t)\,dt$ for $x \in I$
Then $F$ is differentiable on $I$, and

$$\boxed{F'(x) = f(x)}$$

(88)

(2) Let $G$ be any function such that $G'(x) = f(x)$
Then
$$\int_a^b f(x)\,dx = G(b) - G(a)$$

Such a $G$ is called an antiderivative.

Proof:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

(1)
$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h\to 0} \frac{1}{h}\, h \cdot f(c_h)$$

By mean value theorem of integrals

Now, $x \le c_h \le x+h$ so as $h \to 0$, $c \to x$

and we conclude that
$$F'(x) = f(x).$$

(2) If $G$ is any antiderivative of $f$ then
$(G-F)' = 0$, so $G = F + C$, $C$ constant

So $G(b) - G(a) = F(b) + C - F(a) - C = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$

$$= \int_a^b f(t)\,dt$$

$\boxed{M}$

89

Recall: we can think of $\frac{d}{dx}$ as a function of functions

$$\frac{d}{dx}: \{\text{Functions}\} \longrightarrow \{\text{functions}\}$$
$$f \longmapsto f'$$

and so we have shown that the integral is the inverse function

$$\int_a^x : \{\text{functions}\} \longrightarrow \{\text{functions}\}$$
$$f' \longmapsto f$$

So we may write

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x).$$

or $$\int_c^x f'(t)\, dt = f(x)$$

We will use the notation: $\int f(x)\,dx$ to mean some antiderivative of a function, while the notation $\int_a^x f(t)\,dt$ is a fixed antiderivative

Ex: $\int x\, dx = \frac{x^2}{2} + C$ but $\int_1^x t\, dt = \frac{x^2}{2} - \frac{1}{2}$

## Simple integrals

(1) $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$ , $n \neq -1$

(2) $\int \frac{1}{x} \, dx = \ln|x| + C$

(3) $\int e^x \, dx = e^x + C$ $\quad \int a^x \, dx = \frac{1}{\ln a} a^x + C$

(4) $\int \sin x \, dx = -\cos x + C$ $\quad \int \cos x \, dx = \sin x + C$

$\int \sec^2 x \, dx = \tan x$

(5) $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$ $\quad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

## Substitution

Since integration is the inverse of differentiation, we can use our differentiation rules to create integration rules

Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

So, we see that

$$\int f'(g(x)) \cdot g'(x) \, dx = f(g(x))$$

(9)

Typically however we view it as a substitution. That is, if we have

$\int f(g(x)) \cdot g'(x)\, dx$, then we set $u = g(x)$ & hence $du = g'(x)\, dx$ and we can rewrite

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

For definite integrals we have

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Ex: $\displaystyle\int \frac{x}{x^2+1}\, dx = \ln\left(\sqrt{x^2+1}\right) + C$

$\displaystyle\int \frac{\sin(3\ln x)}{x}\, dx = \frac{-1}{3}\cos(3\ln x + C)$

$\displaystyle\int \tan x\, dx = \ln|\sec x| + C$

$\displaystyle\int \frac{1}{a^2+x^2}\, dx - \frac{1}{a}\tan^{-1}(a) + C$

(92)

Can use similar substitutions to generalize our simple integrals

① $\int \sin ax \, dx = \frac{-1}{a} \cos ax + C$ $\qquad \int \cos ax \, dx = \frac{1}{a} \sin ax + C$

② $\int a^{bx} dx = \frac{1}{b \ln a} a^{bx} + C$

③ $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$ $\qquad u = \frac{x}{a}$

Ex: $\int \sin^2 x \, dx$

Double angle formula $\Rightarrow$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\rightarrow$ $\int \sin^2 x \, dx = \frac{1}{2} \int 1 - \cos 2x \, dx = \frac{1}{2}\left(x - \frac{1}{2} \sin 2x\right)$

$= \frac{1}{2}(x - \sin x \cos x) + C$

since $\sin 2x = 2 \sin x \cos x$

Similarly $\int \cos^2 x = \frac{1}{2}(x + \sin x \cos x) + C$

$\int \frac{1}{\cos x} dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx$

Let $u = \sec x + \tan x \Rightarrow du = \sec x \tan x + \sec^2 x \, dx$

$= \sec x (\sec x + \tan x) \, dx$

Hence $\int \frac{1}{\cos x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C$

$= \ln|\sec x + \tan x| + C$

## Integration by Parts

The other main differentiation rule we have is the product rule which tells us

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \left(\frac{d}{dx} f(x)\right)\cdot g(x) + f(x)\cdot\left(\frac{d}{dx} g(x)\right)$$

Integrating both sides we get

$$\int \frac{d}{dx}\left(f(x)\cdot g(x)\right) dx = \int\left(\frac{d}{dx} f(x)\right)\cdot g(x)\, dx + \int f(x)\cdot\left(\frac{d}{dx} g(x)\right) dx$$

$$\Rightarrow \quad f(x)\cdot g(x) = \int g(x)\cdot df(x) + \int f(x)\, dg(x)$$

$$\rightarrow \quad \int f(x)\, dg(x) = f(x)g(x) - \int g(x)\cdot df(x).$$

Typically we rewrite this as

$$\int u\, dv = uv - \int v\, du. \longrightarrow \text{IBP}$$

(94)

In general, we choose $v$ to be something for which we can integrate and $u$ for something which becomes simpler after taking a derivative.

Ex: 1) $\int x e^x \, dx$ $\qquad u = x, \quad dv = e^x dx$

$du = 1 dx \quad v = e^x$

$= xe^x - \int e^x dx = xe^x - e^x + C$

2) $\int x^2 \sin x \, dx$ $\qquad u = x^2 \quad dv = \sin x \, dx$

$du = 2x dx \quad v = -\cos x$

$= -x^2 \cos x - 2\int -x \cos x \, dx$ $\qquad u = x \quad dv = \cos x$

$du = dx \quad v = \sin x$

$= -x^2 \cos x - 2\left( x \sin x - \int \sin x \, dx \right)$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$

3) $\int \ln x \, dx$ $\qquad u = \ln x \quad dv = dx$

$du = \frac{1}{x} dx \quad v = x$

$= x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$