81

Section 4 : Integration

Notation Given a function defined on the naturals $f: \mathbb{N} \to \mathbb{R}$ we define the notation

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + \dots + f(n)$$

Properties:

① $\sum_{i=m}^{n} (A f(i) + B g(i)) = A \cdot \sum_{i=m}^{n} f(i) + B \sum_{i=m}^{n} g(i)$

② $\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^{n} f(i+m)$

③ $\sum_{j=1}^{n} 1 = 1 + 1 + 1 + \dots + 1 = n$

④ $\sum_{j=m}^{n} A = (n-m+1) \cdot A$

Since we typically only sum over the naturals sometimes we will then instead use subscript notation.

That is:

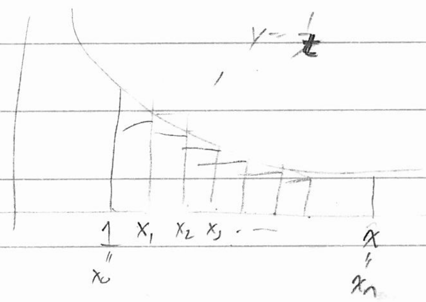
$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \dots + a_n$$

OR:

$$\sum_{i=m}^{n} f(a_i) = f(a_m) + f(a_{m+1}) + \dots + f(a_n)$$

Recall: We defined $\ln(x)$ as the area under the curve $1/t$ from 1 to t.

We can use this sigma notation to approximate $\ln(x)$ that is:



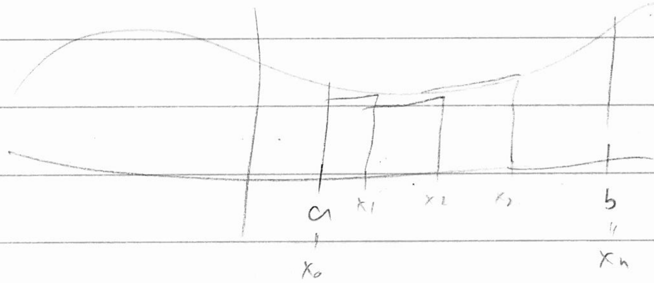
The i^{th} rectangle has a width of $x_i - x_{i-1}$ and a height of $1/x_i$

So if we denote $\Delta x_i = x_i - x_{i-1}$ then the i^{th} rectangle has area $\Delta x_i / x_i$ and we can say

$$\ln(x) \approx \sum_{i=1}^{n} \frac{\Delta x_i}{x_i}$$

and in fact, then $$\ln(x) = \lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} \frac{\Delta(x_i)}{x_i}$$

In general, if we have any curve $y = f(x)$



Then the i^{th} rectangle will have width Δx_i but height $f(x_i)$ and area $\Delta x_i \cdot f(x_i)$ and so

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i) \Delta x_i$$

and $\text{Area} = \lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i) \Delta x_i$ provided the limit exists

These sums are called Riemann Sums

Definition A partition of an interval $[a,b]$ is a set $P = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 \dots < x_n = b$$

For any partition $P = \{x_0, x_1, \dots x_n\}$ of an interval $[a,b]$, we say the upper Riemann sum $U(f,P)$ for the function and the partition P is

$$U(f,P) = \sum_{i=1}^{n} f(x_i) \Delta(x_i)$$

while the lower Riemann sum $L(f,P)$ is

$$L(f,P) = \sum_{i=1}^{n} f(x_{i-1}) \Delta(x_i)$$

Definition If there is exactly one number I such that

$$L(f,P) \leq I \leq U(f,P)$$

for every partition of $[a,b]$ then we say f is ~~integrable~~ on $[a,b]$ and we call I the definite integral of f on $[a,b]$ and write

$$I = \int_a^b f(x)\,dx.$$

Comments: $\int$ is the integral sign and is supposed to resemble an S, as it is the limit of a sum.

The dx takes the place of the Δx_i as we are taking the Δx_i being infinitesmally small. $\longrightarrow$ onto back

The a and b are called the limits of integration with a being the lower limit and b being the upper limit.

$\int_a^b f(x)\,dx$ is the area under the curve $y=f(x)$ from a to b, provided such a thing exists.

The x that is appearing is called a "dummy variable". We may replace it with any other variable. That is

$$\int_a^b f(x)\,dx \quad \text{is the same as} \quad \int_a^b f(t)\,dt$$

Comments about dx: We can view dx as a deliminator of the integral which tells us what we are integrating with respect to much the same way it tells us what we are taking the derivative with respect to in d/dx.

We can also view dx as a new variable called the differential of x which measures how fast x is changing

Therefor if $y = f(x)$ then we can say

$$dy = \frac{dy}{dx}\, dx = f'(x)\, dx$$

That is, y is changing at the rate of $f'(x)$ times the rate that x is changing.

(86)

Properties:

1) $\int_a^a f(x)\,dx = 0$ — the width of our rectangle is 0

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ — our Δx will switch from positive to negative

3) $\int_a^b A f(x) + B g(x)\,dx = A\int_a^b f(x)\,dx + B\int_a^b f(x)\,dx$

— holdover property from summations

4) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ — the area from a to b + the area from b to c = the area from a to c

5) $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$ — the heights may be negative on the LHS but not on the RHS.

The triangle inequality

6) If f is odd $\int_{-a}^{a} f(x)\,dx = 0$

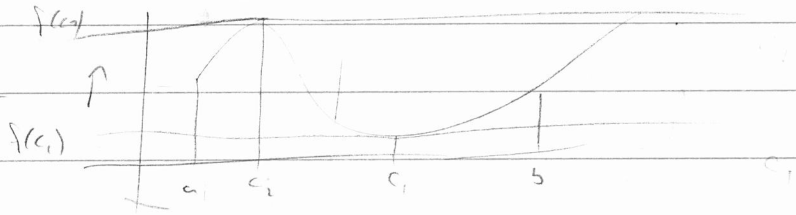
7) if f is even $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

8) The mean value theorem for integrals: If f is continuous on $[a,b]$ then there exists a $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

(87)

Proof:



$$f(c_1)\,(a-b) \le \int_a^b f(x)\,dx \le f(c_2)(a-b)$$

As we increase $c_1 \to c_2$ eventually get equality. ∎

Again, recall that $\ln(x)$ is defined as the area under $1/t$ from 1 to x. So we may write

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

Moreover, we saw that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.

In fact this is true in general.

Fundamental Theorem of Calculus:

Suppose f is a continuous function on an interval I containing a.

(1) Let $F(x) = \int_a^x f(t)\,dt$ for $x \in I$.
Then F is differentiable on I, and

$$\boxed{F'(x) = f(x)}$$