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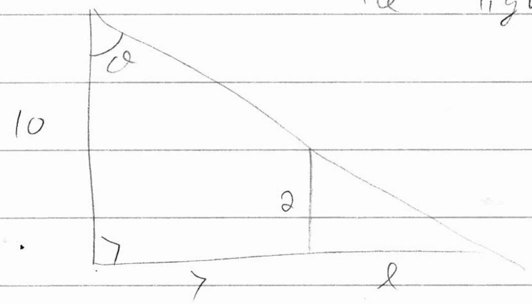
Ex: There is a spotlight at the top of a pole 10 m high. It takes one minute to go from pointing straight down to pointing straight forward. If a 2 m tall man is standing 7 m away from the pole, how fast is his shadow growing, when the light is halfway through its rotation?



$$\frac{d\theta}{dt} = \frac{\pi}{2} \text{ rad/min}$$

$$\tan\theta = \frac{7+\ell}{10}$$

$$\Rightarrow \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{d\ell}{dt}$$

$$\frac{d\ell}{dt} = 2 \cdot \frac{\pi}{2} = \pi$$

Ex: #26 p.221 #35

Indeterminate forms & l'Hôpital's rule

There are many forms of limits that we still don't have a good grasp on solving because they approach something which doesn't make sense i.e. an indeterminate form

(7.1)

Type of indeterminate form	Example
$0/0$	$\lim_{x\to 0} \frac{\sin x}{x}$
∞/∞	$\lim_{x\to 0} \ln(1/x^2) / \cot(x^2)$
$0 \cdot \infty$	$\lim_{x\to 0^+} x \cdot \ln \frac{1}{x}$
$\infty - \infty$	$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$
0^0	$\lim_{x\to 0^+} x^x$
∞^0	$\lim_{x\to (\pi/2)^-} (\tan x)^{\cos x}$
1^∞	$\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x$

These are called indeterminates because only knowing their types do not tell you what the limit is.

examples of types: $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{x}{x^2} = 0$ $\quad \lim_{x\to\infty} \frac{x^2}{x} = \infty$ $\quad \lim_{x\to\infty} \frac{cx}{x} = c$

That is, if the numerator goes to ∞ faster than denominator then the limit is ∞, if the denominator goes to ∞ faster then the limit is 0, while if they both go to ∞ at the same rate, the limit could be anything.

But this hints that maybe derivative could be the answer.

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$0 \cdot \infty$ $\qquad \frac{\infty}{\infty}$

(3) $\lim_{x \to 0^+} x \cdot \ln\left(\frac{1}{x}\right) = \lim_{x \to 0^+} \frac{\ln\left(\frac{1}{x}\right)}{1/x} = \lim_{x \to 0^+} \frac{x \cdot \frac{-1}{x^2}}{-1/x^2} = \lim_{x \to 0^+} x = 0$

(4) $\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$

$= \lim_{x \to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$

(5) $\lim_{x \to 0^+} x^x$ consider $\lim_{x \to 0} \ln(x^x) = \lim_{x \to 0} x \cdot \ln x = \lim_{x \to 0} \frac{\ln x}{1/x}$

$= \lim_{x \to 0} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x \to 0} -x = 0$

So $\lim_{x \to 0^+} x^x = \lim_{x \to 0} e^{\ln(x^x)} = e^{\lim_{x \to 0} \ln(x^x)} = e^0 = 1$

(6) $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

consider

$\lim_{x \to \frac{\pi}{2}^-} \ln(\tan x^{\cos x}) = \lim_{x \to \frac{\pi}{2}} \cos x \cdot \ln(\tan x) = \lim_{x \to \frac{\pi}{2}} \frac{\ln \tan x}{1/\cos x} = \text{"}\frac{\infty}{\infty}\text{"}$

$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\frac{-1}{\cos^2 x} \cdot -\sin x} = \lim_{x \to \frac{\pi}{2}} \frac{1}{\tan x \cdot \sin x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin^2 x} = 0$

$\Rightarrow \lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x} = e^0 = 1$

(7) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ consider $\lim_{x \to \infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x \to \infty} x \cdot \ln\left(1 + \frac{1}{x}\right)$

$= \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} = \lim_{x \to \infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}{-1/x^2} \quad \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = 1$

$\Rightarrow \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

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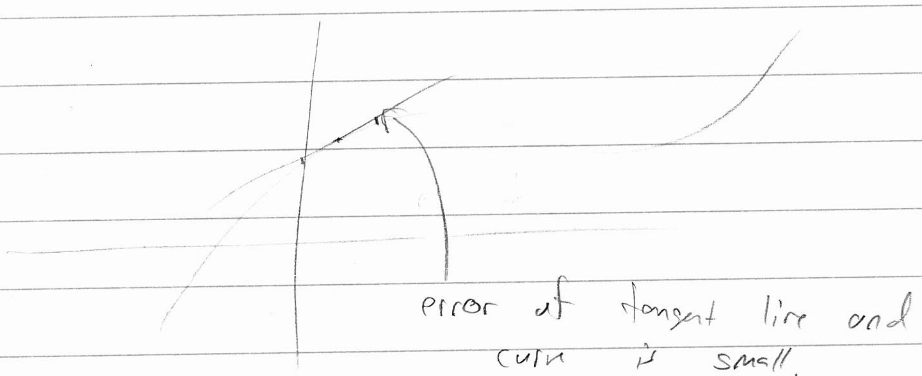
Taylor polynomials

Recall that the tangent line of a function at a point is the line that touches the graph at that point and changes at the same rate as the graph is changin at that point.

Therefore as long as you don't more too far away from your initial point, the tangent line approximates your function.



So therefore we say that

$$f(x) \approx L_a(x) \quad \text{for } x \text{ near } a.$$

where $L_a(x)$ is the tangent line at a

Furtle recall that $L_a(x) = f(a) + f'(a)(x-a)$

(>5)

So, we can say

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \text{ near } a.$$

Use this to approximate $\sin\left(\frac{51\pi}{100}\right)$.

Note that $\frac{101\pi}{100} = \pi + \frac{\pi}{100}$ and we know what happens to sin of π so use $a = \pi$. That is

$$\sin x \approx \sin(\pi + \cos(\pi)(x-\pi) \quad \text{for } x \text{ near } \tfrac{\pi}{2}$$

so that

$$\sin\left(\frac{51\pi}{100}\right) \approx \sin(\pi) + \cos(\pi\left(\frac{101\pi}{100} - \pi\right) = 0 - 1 \cdot \frac{\pi}{100} = \frac{-\pi}{100}$$

Question: How good is this approximation?

If we set $E(x) = f(x) - L_a(x)$, the error function.

Well if $x = a$: Then $E(a) = f(a) - L_a(a) = 0$

First:

Generalized mean value theorem

If f and g are continuous on $[a,b]$ then there exists a $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

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<u>Therefore</u>: If we consider $E(x)$ on the interval $[a, x]$, we get

$$\frac{E(x)}{(x-a)^2} = \frac{E(x) - E(a)}{(x-a)^2 - (a-a)^2} = \frac{E'(c)}{2(c-a)} \quad \text{for some } c \in (a, x)$$

$$= \frac{1}{2}\frac{f'(c) - f'(a)}{c-a} = \frac{1}{2} f''(s) \quad \text{for some } s \in (a, c)$$

That is, if x is near a, then
$$E(x) = \frac{1}{2} f''(s)\,(x-a)^2 \quad \text{for some } s \text{ near } a.$$

Hence we can say that

$$|E(x)| \le \frac{1}{2} \max_{s \text{ near } a} \left(|f''(s)|\right) \cdot (x-a)^2$$

For example: we saw that
$$\sin\left(\frac{101\pi}{100}\right) \approx L_\pi\left(\frac{101\pi}{100}\right) = -\frac{\pi}{100}$$

And so
$$\left|E\left(\frac{101\pi}{100}\right)\right| \le \frac{1}{2} \max_{s \text{ near } \frac{\pi}{2}} \left(|\sin s|\right) \left(\frac{101\pi}{100} - \pi\right)^2$$
$$\le \frac{1}{2}\left(\frac{\pi}{100}\right)^2$$

That is, we conclude that
$$\left|\sin\left(\frac{101\pi}{100}\right) - \frac{-\pi}{100}\right| \le \frac{1}{2}\left(\frac{\pi}{100}\right)^2 = 0.0004934$$

And we can say that $\sin\left(\frac{101\pi}{100}\right)$ is equal to $\frac{\pi}{100}$ to three decimal points

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In reality:

$$\sin\left(\frac{101\pi}{100}\right) = -0.0314107590\overline{7}$$
$$-\frac{\pi}{100} = -0.031415926$$
$$\left|\sin\left(\frac{101\pi}{100}\right) - \frac{-\pi}{100}\right| = 0.00000516\overline{7}\ldots$$

So, we see that the error between the curve and the tangent line is controlled by the second derivative. So can we use the second derivative to get a better approximation?

Indeed, if we define

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

to be the Taylor polynomial of degree 2 of f at a, then we can do the same process to show that

$$|E_2(x)| = |f(x) - P_2(x)| \leq \frac{1}{6} \max_{s \text{ near } a} |f'''(s)| (x-a)^3$$

That is, the error is controlled by the third derivative.

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Can we go further? Define

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where for an natural number m, $m! = m \cdot (m-1) \cdot (m-2) \cdots 2 \cdot 1$ and $f^{(m)}(a)$ is the m^{th} derivative of f at a.

This is called the Taylor polynomial of degree n of f at a.

Further we have that

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{1}{(n+1)!} \max_{s \text{ near } a} |f^{(n+1)}(s)| \, (x-a)^{n+1}$$

Note: if $a=0$, we sometimes call it just a Maclaurin series

We can then use this estimate values of functions to arbitrary precision

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Ex: Approximate $\cos(1)$ to an error of $1/5000$

We want to write
$$f(x) = \cos(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Such that $|E_n(1)| \leq \frac{1}{(n+1)!} \max_s |f^{(n+1)}(s)| \, (1-a)^{n+1} \leq \frac{1}{5000}$

Use $a = 0$, and find higher order derivatives of $\cos x$
$f(x) = \cos x$, $f'(x) = -\sin x$ $f''(x) = -\cos x$ $f'''(x) = \sin x$ $f^{(4)}(x) = \cos x$
So repeates. Hence $|f^n(s)| \leq 1$

So find an n s.t. $|E_n(1)| \leq \frac{1}{(n+1)!} \leq \frac{1}{5000}$
$n = 6$ works

So, up to a possible error of $\frac{1}{5000}$ we have

$$\cos(1) \approx \cos(0) - \sin(0)(1-0) - \frac{\cos(0)}{2}(1-0)^2 + \frac{\sin(0)}{6}(1-0)^3 + \frac{\cos(0)}{24}(1-0)^4$$
$$- \frac{\sin(0)}{120}(1-0)^5 - \frac{\cos(0)}{840}(1-0)^6$$
$$= 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{840} = 0.540476\dots$$

While in reality: $\cos(1) = 0.5403023\dots$

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We can see here that cos x would have a nice Taylor series of degree n at 0. since it's derivatives repeat after a while

That is

$f(x)$	$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$
$\cos x$	$\cos(0)=1$	$-\sin(0)=0$	$-\cos(0)=-1$	$\sin(0)=0$	$\cos(0)=1$
$\sin x$	$\sin(0)=0$	$\cos(0)=1$	$-\sin(0)=0$	$-\cos(0)=-1$	$\sin(0)=0$
e^x	$e^0=1$	$e^0=1$	$e^0=1$	$e^0=1$	$e^0=1$

So $$\cos(x) \approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\dots+\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) \approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\dots\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$e^x \approx 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\dots+\frac{x^n}{n!}$$

$$\ln(1+x) \approx x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots+(-1)^{n-1}\frac{x^n}{n}$$

$$\frac{1}{1-x} \approx 1+x+x^2+x^3+x^4+x^5+\dots+x^n$$

Notice: cos only has even powers and is an even function will sin only has odd powers and is an odd function.