60)

Curve sketching

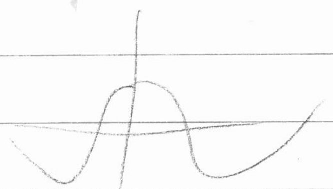
We can use the derivative to get a good idea of what the graph of a function looks like. That is, we can create a rough sketch of the graph that shows the most important aspects

Important aspects:

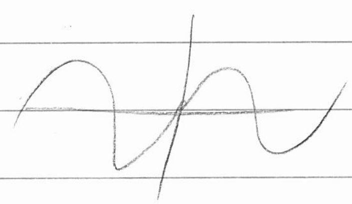
① Symmetry We say a function is even if it satisfies $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

If f is even then its graph is symmetric about the y-axis whereas if f is odd it is antisymmetric.

$\cos x$ is even



$\sin x$ is odd



② Zeros: Where does the graph cross the x-axis? this will be where the numerator is zero.

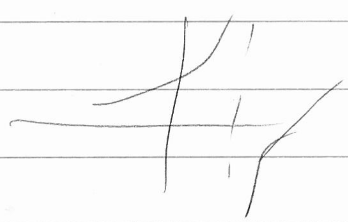
G1)

③ Asymptotes:

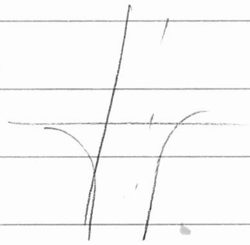
Does the graph blow tend to ∞ or $-\infty$ anywhere? This will be a vertical asymptote and will happen when the denominator is 0.

Does it tend to ∞ or $-\infty$ on the left or right of the asymptote?

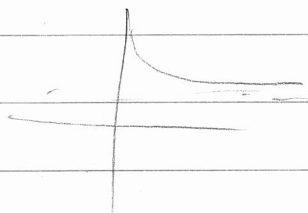
or



or



What happens as x tends to ∞ or $-\infty$? These will be horizontal asymptotes (if they exist). Do they approach from above, below or neither?



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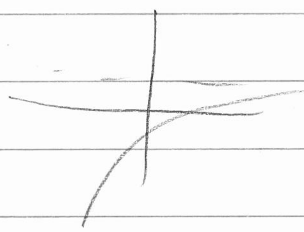

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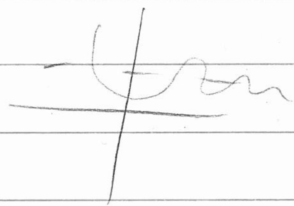
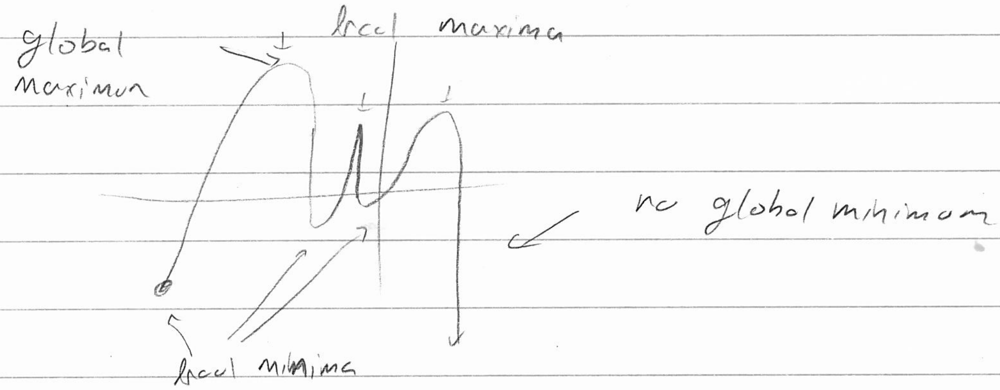


④ Critical points, singular points and endpoints

The critical points are where the derivative is zero and singular points are where the derivative does not exist.

Recall, we said that if a function achieves its maximum and/or its minimum, it must be at a critical point, a singular point or an endpoint.

We define a <u>local maximum</u> or <u>minimum</u> that is larger or smaller than everything near it.

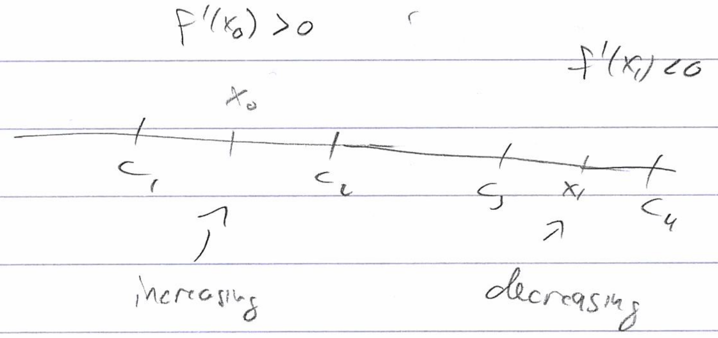


Similarly, if a function has a local maximum or minimum then it must be a critical point, a singular point or an endpoint.

(5) Intervals of increasing / decreasing

The derivative can not be zero in between two consecutive critical points. Therefore if the derivative is positive anywhere between two critical points, then it is positive on the whole interval and thus increasing on the whole interval. Likewise if the derivative is negative then it is decreasing.

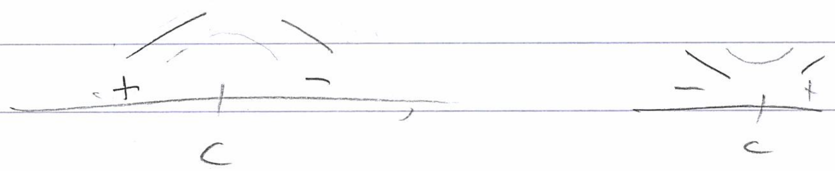
i.e.



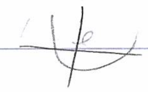
6 Local maxima and minima

If a function goes from increasing to decreasing at a critical point then it is a local maximum. If it goes from decreasing to increasing then it is a local minimum

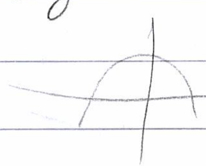
i.e.



7 The second derivative and intervals of concavity

If the second derivative is positive then this means that the derivative is increasing and thus we say it is concave up. 

Likewise if the second derivative is negative then the derivative is decreasing and thus will be concave down



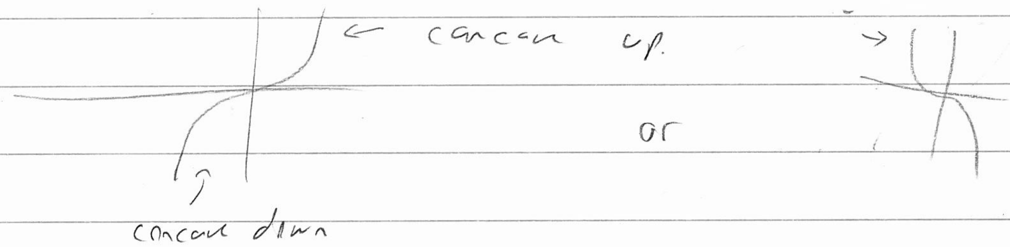
These are the intervals of concavity.

We can also use the second derivative to test critical points for being maximum or minimum

If c is a critical point and $f''(c) > 0$ then the f' is increasing i.e. going from $-$ to $+$ and so is a minimum.
Likewise if c is a critical point and $f''(c) < 0$ then c is a maximum.

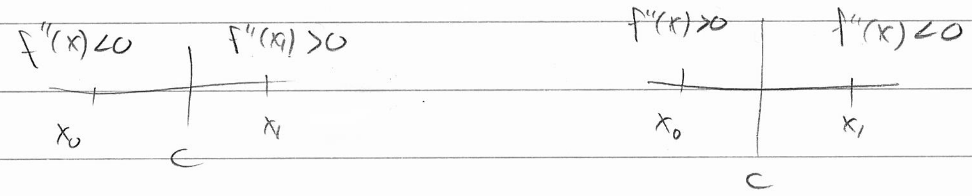
(8) Inflection points

An inflection point is a point where the graph switches concavity.

i.e 

If c is an inflection point then $f''(c) = 0$.
Caution: If $f''(c) = 0$ this is not necessarily an inflection point!!!

For c to be an inflection point we need f'' to change sign at c.

i.e. 

(65)

Further caution! $f''(c) = 0 \not\Rightarrow f'(c) = 0$

That is, inflection points need not be critical points!

With all this now, let's sketch some curves:

$f(x) = \dfrac{x^2-1}{x^2-4}$ $\quad$ $f(-x) = f(x)$ so even

V.A: $x = \pm 2$ $\quad$ H.A: $x = 1$

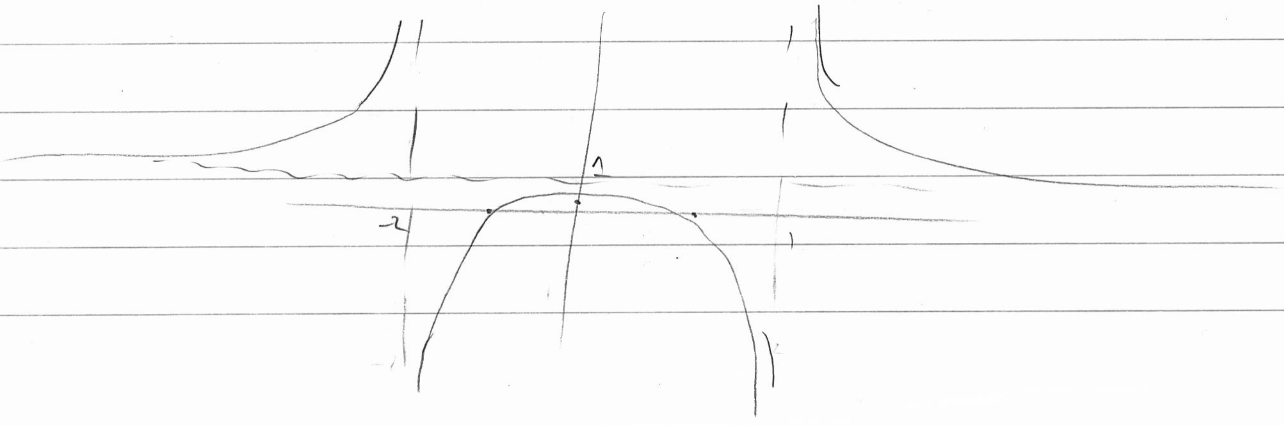
x-intercepts $x = \pm 1$ $\quad$ $\lim_{x \to -2^-} = \infty$

$f'(x) = \dfrac{-6x}{(x^2-4)^2}$ $\quad$ critical points: $x = 0$ $\quad$ $\lim_{x \to -2^+} = -\infty$

singular points: $x = \pm 2$ $\quad$ $\lim_{x \to 2^-} = -\infty$

$f''(x) = \dfrac{6(3x^2+4)}{(x^2-4)^3}$ $\quad$ no inflection points $\quad$ $\lim_{x \to 2^+} = \infty$

x		-2		0		2	
f'	+	und	+	0	−	und	−
f''	+	und	−	−	−	und	+
f	↗	und	↗	max	↘	und	↘
	∪		∩	$\frac{1}{4}$	∩		∪



(66)

$f(x) = (x^2-1)^{2/3}$

zero: $x = \pm 1$

V.A.: None H.A.: None

$f(x) = f(-x)$ even

$f'(x) = \dfrac{4x}{3(x^2-1)^{1/3}}$ C.p. $x = 0$, S.p. $= \pm 1$

$f''(x) = \dfrac{4}{9}\dfrac{(x^2-3)}{(x^2-1)^{4/3}}$ possible inflection point $\pm\sqrt{3}$

	CP		SP		PIPS	
x	0		1		$\sqrt{3}$	
f'	0	−	und	+	+	+
f''	−	−	und	−	0	+
f	max	↗	und	↗	IP	↗
		∩		∩		∪