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Differential Equations

We have seen that if $f(x) = e^x$ then
$$\frac{d}{dx} f(x) = e^x = f(x)$$

We call these types of equations (i.e. $\frac{d}{dx} f(x) = f(x)$) differential equations as they relate the derivative of the equation to itself.

We often want to find all possible functions that satisfy differential equations.

That is, what are the other functions (if any) that satisfy $\frac{d}{dx} f(x) = f(x)$

Well, if we had such a function then let's compare it to e^x. That is, what is $f(x)/e^x$? All we know about $f(x)$ is its derivative, so lets do that:

$$\frac{d}{dx}\left(\frac{f(x)}{e^x}\right) = \frac{\left(\frac{d}{dx}(f(x))\right)\cdot e^x - f(x)\frac{d}{dx}e^x}{e^{2x}} = \frac{f(x)\, e^x - f(x)\, e^x}{e^{2x}} = 0$$

so $\frac{f(x)}{e^x} = C \iff f(x) = C \cdot e^x$

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Often, we set $y = f(x)$ and then simplify the differential equation

$\frac{d}{dx}(f(x)) = f(x)$ to $y' = y$

with the understand that y is a function of x and we are taking its derivative with respect to x.

i.e. we write $y' = y \Rightarrow y = C \cdot e^x$

So, what about the slightly more complicated differential equation

$$y' = ry$$

First gues: should be related to e^x but the derivative should have a multiplication by r in it.

So try $y = e^{rx}$

Indeed $y' = r \cdot e^{rx} = r \cdot y$.

So, what others? Compare such a y with e^{rx} to find that

$$\frac{d}{dx}\left(\frac{y}{e^{rx}}\right) = \frac{y' \cdot e^{rx} - rye^{rx}}{e^{2rx}} = \frac{rye^{rx} - rye^{rx}}{e^{2rx}} = 0$$

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So, we conclude that

$$y' = ry \implies y = C \cdot e^{rx}$$

Often times differential equations will come will <u>initial conditions</u>
Thes initial conditions will determine what our C value will be.

Ex: $y' = 3y$, $y(0) = 5$ $\implies$ $y = C \cdot e^{3x}$

But $5 = y(0) = C \cdot e^{3 \cdot 0} = C$

So, $y = 5 \cdot e^{3x}$

Need not be $y(0)$:

$y' = 20y$, $y(2) = 7$ $\implies$ $y = C \cdot e^{20x}$

But $7 = y(2) = C \cdot e^{40} \implies C = \frac{7}{e^{40}}$

& $y = \frac{7}{e^{40}} e^{20x} = 7 \cdot e^{20x - 40}$

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Second order linear differential equations

Differential equations of the form

$$ay'' + by' + cy = 0 \quad , \quad a \neq 0$$

are called second order linear differential equations

How to solve? Well, again we are relating a function and its derivative, so a first guess would be $y = e^{rx}$.

Try: $a(e^{rx})'' + b(e^{rx})' + c(e^{rx})$

$= ar^2 e^{rx} + br \cdot e^{rx} + ce^{rx}$

$= (ar^2 + br + c) e^{rx}$

So for this to be 0, we need

$ar^2 + br + c = 0$.

Use quadratic formula to surmise that

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

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<u>Case 1</u>: $b^2 - 4ac > 0$

Then we get two real numbers r_1, r_2 such that $e^{r_1 x}$ and $e^{r_2 x}$ are solutions

So we can now see that everything of the form

$$y = Ae^{r_1 x} + Be^{r_2 x}$$

is a solution to the differential equation

Indeed: $y' = Ae^{r_1 x} \cdot r_1 + Be^{r_2 x} \cdot r_2$
$y'' = Ae^{r_1 x} \cdot r_1^2 + Be^{r_2 x} \cdot r_2^2$

So,
$$ay'' + by' + cy' = Ae^{r_1 x}(ar_1^2 + br_1 + c) + Be^{r_2 x}(ar_2^2 + br_1 + c) = 0.$$

In fact these are the only solutions. That is,

$$ay'' + by' + cy = 0 \implies y = Ae^{r_1 x} + Be^{r_2 x}$$

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Sketch of a proof of this:

Set $y = e^{r_1 x} \cdot u(x)$.
Then show that u must satisfy the differential equation

$$u'' = (r_2 - r_1) u'$$

Set $u' = v$ so that v satisfies the differential equation

$$v' = (r_2 - r_1) v \quad \text{an so} \quad v = Ce^{(r_2 - r_1)x}$$

And thus $u = De^{(r_2 - r_1)x}$
and $y = De^{r_2 x}$.

Case 2: $b^2 - 4ac = 0$

Then there is only one value of r; and we get $y = e^{rx}$ is a solution.

Suppose $y = e^{rx} \cdot u(x)$ is a different solution. Then

$$y' = e^{rx}(u + ru') \qquad y'' = e^{rx}(u'' + 2ru + r^2 u).$$

Thus:

$$0 = ay'' + by' + cy = e^{rx}(au'' + (2ar + b)u' + ar^2 + br + c)$$

$$= e^{rx} \, au''$$

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Hence $u'' = 0$ so $u' = B$

& $(u - Bx)' = 0$ so $u = A + Bx$

Therefore, we conclude that if $b^2 - 4ac = 0$ we get:

$$ay'' + by' + cy = 0 \Rightarrow y = (A + Bx)e^{rx} = Ae^{rx} + xBe^{rx}$$

Case 3: $b^2 - 4ac < 0$.

If we denote i to be value such that $i^2 = -1$ (i.e. the imaginery unit), then we get the two solutions will be

$$r_1 = b + i\omega \qquad r_2 = b - i\omega$$

And by the same as case 2, we have the general solution will be

$$y = Ae^{(b+i\omega)x} + Be^{(b-i\omega)x}$$

However, we have an identity

$$e^{ix} = \cos x + i\sin x \quad \text{(will prove later)}$$

And so we can simplify down to:

$$y = Ae^{bt}\cos(\omega t) + Be^{bt}\sin(\omega t)$$

Examples: Find the general solution to

$$y'' + y' - 6y = 0 \Rightarrow y = Ae^{-3x} + Be^{2x}$$
$$y'' + y' + y = 0 \Rightarrow y = Ae^{x/2}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + Be^{x/2}\sin\left(\tfrac{\sqrt{3}}{2}x\right)$$
$$y'' + y = 0 \Rightarrow y = A\cos x + B\sin x$$

For second order linear differentials we need two initial conditions to find just one solution:

$$y'' - y = 0 \Rightarrow y = \frac{e^{x} + e^{-x}}{2} = \cosh$$
$y(0) = 1$
$y'(0) = 0$

If change to
$y(0) = 0$
$y'(0) = 1$
$$\Rightarrow y = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

Nonhomegeneous differential linear equations

A non homogeneous linear equation is one where the right hand side is not zero.

ex: $y'' + y' - 6y = x^2 + 2$

Theorem If y_p is a solution to the differential equation $ay'' + by' + cy = f(x)$ & y_h is a solution to $ay'' + by' + cy = 0$ then $y_p + y_h$ is a solution to $ay'' + by' + cy = 0$

Proof: $a(y_p + y_h)'' + b(y_p + y_h)' + c(y_p + y_h) = ay_p'' + by_p' + cy_p + ay_h'' + by_h' + cy_h$
$= f(x) + 0 = f(x)$

So, to solve a non homogeneous differential linear equation, it suffices to find the general homogeneous solution, y_h, and one particular solution y_p.

In general finding a particular solution is very hard. However, we will only worry about two special cases

$$y'' + y' + y = 2x^2 + 3x + 2$$
$$\Rightarrow y = Ae^{x/2}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + Be^{x/2}\sin\left(\tfrac{\sqrt{3}}{2}x\right) + 2x^2 - x - 1$$

$$y'' + 4y = \sin x - 2\cos x$$
$$\Rightarrow y = A\sin 2x + B\cos 2x - \tfrac{1}{3}\sin x + \tfrac{2}{3}\cos x$$
$$y(0) = 0 \quad y'(0) = 2 \Rightarrow y = \tfrac{7}{6}\sin 2x - \tfrac{2}{3}\cos 2x - \tfrac{1}{3}\sin x + \tfrac{2}{3}\cos x$$

Comments on $\cosh x$ & $\sinh x$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x$$

Other interesting properties similar to sin & cos.
See p. 201 - 205.