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<u>Question:</u> What is $\lim_{x \to 0^+} \ln x$? What is $\ln 1$

What is $\lim_{x \to \infty} \ln x$?

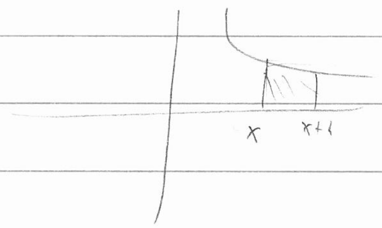
<u>Theorem</u>: for $x > 0$ $\frac{d}{dx} \ln x = \frac{1}{x}$

<u>Proof</u>: $\frac{d}{dx} \ln x = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$

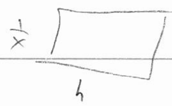
What is $\ln(x+h) - \ln(x)$? The area under $1/t$ between $x+h$ & x

That is

$\ln(x+h) - \ln x =$

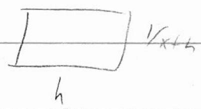


So $\ln(x+h) - \ln x \leq$ the area of the rectangle



$= \frac{h}{x}$

& $\ln(x+h) - \ln x \geq$ the area of the rectangle



$= \frac{h}{x+h}$

So $\frac{1}{x} = \lim_{h \to 0} \frac{h/(x+h)}{h} \leq \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} \leq \lim_{h \to 0} \frac{h/x}{h} = \frac{1}{x}$

☑

<u>Properties of natural logarithm</u>

(1) $\ln(xy) = \ln x + \ln y$

(2) $\ln\left(\frac{1}{x}\right) = -\ln x$

(3) $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln y$

(4) $\ln(x^r) = r \ln x$

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Proof: (1) Fix y and consider the function $\ln xy - \ln x$.
What is its derivative?

$$\frac{d}{dx}(\ln xy - \ln x) = \frac{y}{xy} - \frac{1}{x} = 0$$

Thus $\ln xy - \ln x = C$, some constant.
But which constant? Since this is true for all x, it must be true for $x = 1$. That is

$$C = \ln(1 \cdot y) - \ln 1 = \ln y$$

i.e. $\ln xy - \ln x = \ln y$ or $\ln xy = \ln x + \ln y$.

(2) Again $\frac{d}{dx}\left(\ln \frac{1}{x} + \ln x\right) = \frac{1}{1/x} \cdot \frac{-1}{x^2} + \frac{1}{x} = 0$

$\Rightarrow$ $\ln \frac{1}{x} + \ln x = C$ setting $x = 1 \Rightarrow C = 0$

So $\ln \frac{1}{x} = -\ln x$

(3). $\ln\left(\frac{x}{y}\right) = \ln\left(x \cdot \frac{1}{y}\right) = \ln x + \ln \frac{1}{y} = \ln x - \ln y$

(4) $\ln(x^n) = \ln(x \cdots x) = \ln(x) + \ln(x) + \dots + \ln x = n \cdot \ln x$

$m \ln(x^{n/m}) = \ln(x^{n/m \cdot m}) = \ln(x^n) = n \ln x$

$\Rightarrow \ln(x^{n/m}) = \frac{n}{m} \ln x$

Then for any real r, Find r_n rational such that $r_n \to r$ then $\ln(x^{r_n}) \to \ln(x^r)$

But $\ln(x^{r_n}) = r_n \ln x \to r \cdot \ln x$

So $\ln(x^r) = r \cdot \ln x$

Now we may concretely answer the question $\lim_{x\to\infty} \ln x$. As $x \to 0^+$ then $1/x \to \infty$ and so

$$\lim_{x\to\infty} \ln x = \lim_{x\to 0^+} \ln \frac{1}{x} = -\lim_{x\to 0^+} \ln x = -\infty.$$

The exponential function

Definition We define the exponential function $\exp(x)$ as the inverse to $\ln x$.

i.e. $y = \exp(x) \iff x = \ln y \quad (y > 0)$

Thus: $\ln(\exp x) = x$ & $\exp(\ln x) = x$.

Properties: (1) $(\exp x)^r = \exp(rx)$
(2) $\exp(x+y) = \exp(x) \cdot \exp(y)$
(3) $\exp(-x) = 1/\exp(x)$
(4) $\exp(x-y) = \exp x / \exp y$.

What does property 1 imply?
Inputting $x = 1$, we get that

$$\exp(r) = (\exp(1))^r.$$

So if we define $e = \exp(1)$ then we see that

$$\boxed{\exp(x) = e^x}$$

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Hence also we have $\ln(x) = \log_e x$.

What is e? $e = 2.7\,1828\,1828\,45\,90\,45\ldots$
called Euler's number

<u>Question:</u> What is $\frac{d}{dx} e^x$?

Use implicit differentiation $y = e^x \Rightarrow x = \ln y$
so $1 = \frac{1}{y} \cdot \frac{dy}{dx}$ i.e. $\frac{dy}{dx} = y = e^x$

Therefore, we get one of the most important equations in calculus

$$\boxed{\frac{d}{dx} e^x = e^x}$$

We can use this to calculate the derivative of all exponentials and logarithms

Note that $a^x = e^{\ln(a^x)} = e^{x \cdot \ln a}$

Therefore by change rule we have

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \cdot \ln a} = e^{x \cdot \ln a} \cdot \ln a$$

$$= a^x \cdot \ln a.$$

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Finally, for logarithms!

$$y = \log_a x \implies x = a^y$$

So $\quad 1 = \frac{d}{dx} a^y = a^y \cdot \ln a \cdot \frac{dy}{dx}$

$$\rightarrow \frac{dy}{dx} = \frac{1}{a^y \cdot \ln a} = \frac{1}{x \cdot \ln a}$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}$$

So, we can define all exponentials and logarithms and their derivatives in terms of e, but what is e?

<u>Theorem</u> $\quad e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$

<u>Proof</u> $\ln\left(\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n\right) = \lim_{n \to \infty} \ln\left(\left(1 + \frac{x}{n}\right)^n\right)$

$= \lim_{n \to \infty} n \cdot \ln\left(1 + \frac{x}{n}\right) = x \cdot \lim_{n \to \infty} \frac{\ln\left(1 + \frac{x}{n}\right)}{x/n}$ $\quad$ let $h = \frac{x}{n}$ then $h \to 0$ as $n \to \infty$

$= x \cdot \lim_{h \to 0} \frac{\ln(1+h)}{h} = x \cdot \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h}$

$$= x \left(\frac{d}{dx} \ln x\right)\Big|_{x=1} = x \quad \blacksquare$$

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So, $e^1 = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

n	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.5937..
100	2.70481
1000	2.71692
10000	2.7181459
⋮	
∞	2.718281828459045..... $= e$

Growth rates of exp and ln:

① exp grows faster than any power.

$$\lim_{x\to\infty} \frac{x^a}{e^x} = 0$$

② exp decays faster than any power

$$\lim_{x\to-\infty} |x|^a e^x = 0$$

③ ln grows slower than any power

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = 0$$

④ ln decays slower than any power

$$\lim_{x\to 0^+} x^a \cdot \ln x = 0$$

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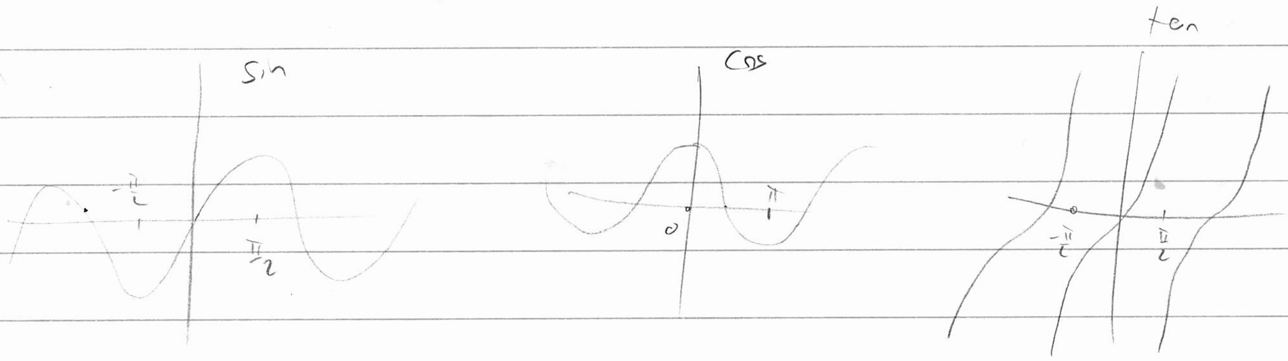
Inverse trig functions and their derivatives

The trigonometric functions, (sin, cos, tan) all fail the horizontal line test and so are not one-to-one. So in order to talk about their inverses we must restrict their domains



So we define the inverse trig functions as the unique values in those ranges

$y = \sin^{-1} x \iff x = \sin(y) \ \& \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$y = \cos^{-1} x \iff x = \cos y \ \& \ 0 \leq y \leq \pi$

$y = \tan^{-1} x \iff x = \tan y \ \& \ -\frac{\pi}{2} < y \leq \pi/2$

We read these as arcsin, arccos & arctan.

	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$
Range	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[0, \pi]$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
at $\frac{-1}{\sqrt{2}}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$	

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Note that we have the identity

$$\cos x = \sin\left(\tfrac{\pi}{2} - x\right)$$

Therefore $y = \cos^{-1} x \iff \cos y = x$

$$\iff \sin\left(\tfrac{\pi}{2} - y\right) = x$$
$$\iff \tfrac{\pi}{2} - y = \sin^{-1} x$$
$$\iff y = \tfrac{\pi}{2} - \sin^{-1} x$$

That is, $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$

Let's find their derivatives!

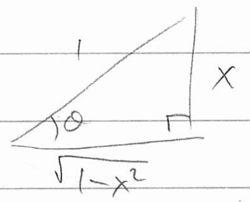
(1) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

proof Set $y = \sin^{-1} x$, so $\sin y = x$

So, $\cos y \frac{dy}{dx} = x \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(\sin^{-1} x)}$

What is $\cos(\sin^{-1} x)$?

Create triangle



So $\sin^{-1} x = \theta$ & $\cos\theta = \sqrt{1-x^2}$

So $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

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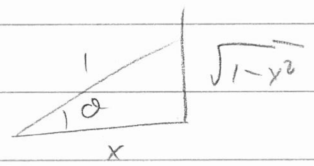
(2) $\dfrac{d}{dx}\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$

proof: Again: $y=\cos^{-1}x \Rightarrow x=\cos y$

so $1 = \dfrac{dy}{dx}\cdot -\sin y$

& $\dfrac{dy}{dx} = \dfrac{-1}{\sin\left(\cos^{-1}x\right)}$



$= \dfrac{-1}{\sqrt{1-x^2}}$

Easier! Use $\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$ to

get $\dfrac{d}{dx}\cos^{-1}x = \dfrac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1}x\right) = -\dfrac{d}{dx}\sin^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$

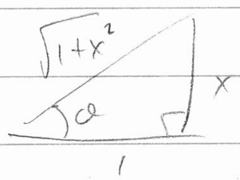
(3) $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$

proof: $y = \tan^{-1}x \Rightarrow \tan y = x$

$\Rightarrow 1 = \sec^2 y\cdot\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\left(\sec^2\left(\tan^{-1}(x)\right)\right)} = \left(\cos\left(\tan^{-1}x\right)\right)^2$

Triangle:



$\cos\left(\tan^{-1}x\right) = \cos\theta = \dfrac{1}{\sqrt{1+x^2}}$

$\rightarrow \dfrac{d}{dx}\tan^{-1}x = \left(\cos\left(\tan^{-1}x\right)\right)^2 = \dfrac{1}{1+x^2}$

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What is $\sec^{-1} x$ in terms of $\cos x$?

$$y = \sec^{-1} x \iff \sec y = x \iff \frac{1}{\cos y} = x$$
$$\iff \cos y = \frac{1}{x} \iff y = \cos^{-1} \frac{1}{x}$$

So $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$ for $|x| \geq 1$

explain

(4) $$\frac{d}{dx} \sec^{-1} x = \frac{1}{x^2 \sqrt{1-\frac{1}{x^2}}} \overset{\text{exercise}}{=} \frac{1}{|x| \sqrt{x^2-1}}$$

Proof $$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{d}{dx}\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$$

Similarly $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$ & $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$

and

(5) $$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x^2 \sqrt{1-\frac{1}{x^2}}} = \frac{-1}{|x| \sqrt{x^2-1}}$$

(6) $$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{x^2\left(1+x^2\right)}$$