# Section 3: Complicated Functions

## Inverse functions:

**Definition** A function is <u>one-to-one</u> if for every possible output you have one input. i.e.

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

Ex: $f(x) = x^3$ is one-to-one but $f(x) = x^2$ is not

Note: one-to-one functions pass the horizontal line test

**Definition** If $f$ is one-to-one then it has an <u>inverse function</u>, denoted $f^{-1}$. The value of $f^{-1}(x)$ is the <u>unique</u> number $y$ in the domain of $f$ for which $f(y) = x$.

i.e. $$y = f^{-1}(x) \iff x = f(y)$$

Ex: if $f(x) = x^3$ then $f^{-1}(x) = x^{1/3}$

If $f$ is not one-to-one then we can sometimes reduce its domain to make it one-to-one and thus it will have an inverse on this domain

Ex $f: \mathbb{R} \to [0,\infty)$, $x \mapsto x^2$ is not one-to-one but

$g: [0,\infty) \to [0,\infty)$, $x \mapsto x^2$ is one-to-one and has inverse $g^{-1}(x) = \sqrt{x}$

Whereas $h: (-\infty, 0] \to [0,\infty)$, $x \mapsto x^2$ is one-to-one with inverse $h(x) = -\sqrt{x}$.

Properties of inverse functions

1. $y = f^{-1}(x) \iff x = f(y)$
2. the domain of $f^{-1}$ is the range of $f$
3. the range of $f^{-1}$ is the domain of $f$
4. $f^{-1}(f(x)) = x$ for all $x$ in the domain of $f$
5. $f(f^{-1}(x)) = x$ for all $x$ in the domain of $f^{-1}$ (or range of $f$)
6. $(f^{-1})^{-1}(x) = f(x)$ for all $x$ in the domain of $f$
7. The graph of $f^{-1}$ is the reflection of $f$ in the line $x=y$.

The derivative:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Exponential function:

An exponential function is a function of the form

$$f(x) = a^x$$

for some $a$.

Recall: $a^0 = 1$

If $n$ and $m$ are positive integers then $a^n = (\overbrace{a \cdots a}^{n \text{ times}})$

$a^{-n} = 1/a^n = 1/(a \cdot a \cdots a)$

$a^{m/n} = \sqrt[n]{a^m}$

For any real number $r$, we can find a sequence of rational numbers $r_1, r_2, r_3 \dots$ such that $\lim_{n\to\infty} r_n = r$ and then we define

$$a^r = \lim_{n\to\infty} a^{r_n}$$

<u>Ex:</u> What is $2^\pi$ ?

Well $\pi = 3.1415926535\dots$

So set $r_1 = 3$ $\quad r_2 = \frac{31}{10}$ $\quad r_3 = \frac{314}{100}, \dots$ and

calculate $2^3 = 8$, $\quad 2^{3.1} = 8.574$ $\quad 2^{3.14} = 8.8152 \quad \dots$

get

$$2^\pi = 8.824977\dots$$

<u>Laws of exponents</u>

If $a, b > 0$ and $x, y \in \mathbb{R}$ then

1. $a^0 = 1$
2. $a^x a^y = a^{x+y}$
3. $a^{-x} = \frac{1}{a^x}$
4. $a^x / a^y = a^{x-y}$
5. $(a^x)^y = a^{xy}$
6. $a^x b^x = (ab)^x$

What is $\lim_{x\to\infty} a^x$, $\quad \lim_{x\to-\infty} a^x$ ?

Depends on $a$: $\quad$ if $a > 1$ get $0$ & $\infty$

if $0 < a < 1$ get $\infty$ & $0$

if $a = 1$ get $1$ & $1$

if $a = 0$ get $0$ & $0$.

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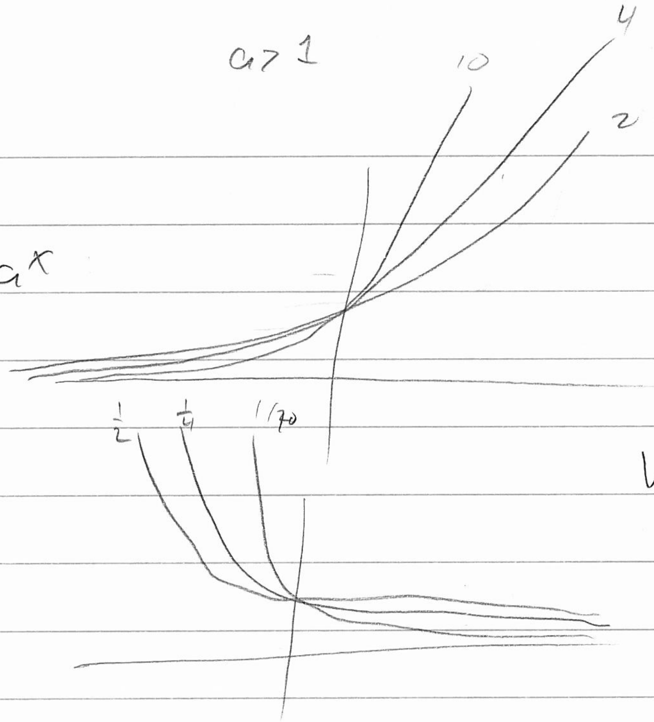
Graph of $y = a^x$




But how fast is it changing?

What is $\frac{d}{dx} a^x$ ??

## Logarithms

Since $f(x) = a^x$ is a one-to-one function, we can take its inverse.

Definition: If $a > 0$ & $a \neq 1$, the function $f(x) = \log_a x$, called the logarithm of $x$ to the base $a$, is the inverse to the function '$a^x$'. i.e.

$$y = \log_a x \iff x = a^y$$

Domain of $\log_a x$ is $(0, \infty)$

Alternative we may see that

$$\log_a (a^x) = x \quad \text{for all } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for all } x > 0.$$

That is, $\log_a$ "cancels" exponentiation of $a$.

## Laws of logs

① $\log_a 1 = 0$

② $\log_a xy = \log_a x + \log_a y$

③ $\log_a \left(\frac{1}{x}\right) = -\log x$

④ $\log_a \left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

⑤ $\log_a (x^y) = y \log_a x$

⑥ $\log_a x = \dfrac{\log_b x}{\log_b a}$

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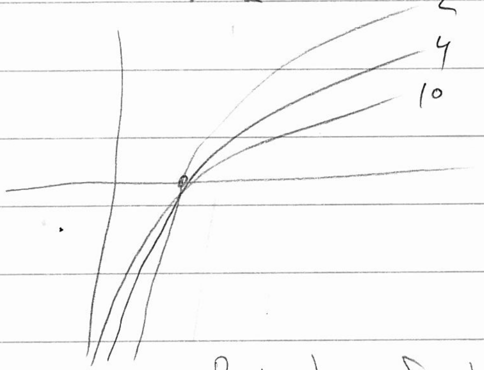
What is $\log_a x$ at its boundraries? Depends on $a$

$a > 1$ $\quad \lim_{x \to 0^+} \log_a x = -\infty \qquad \lim_{x \to \infty} \log_a x = \infty$

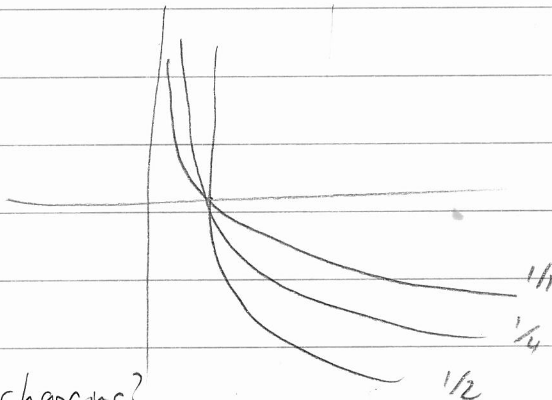
$0 < a < 1$ $\quad \lim_{x \to 0^+} \log_a x = \infty \qquad \lim_{x \to \infty} \log_a x = -\infty$

Graphs!

$a > 1$




$a < 1$

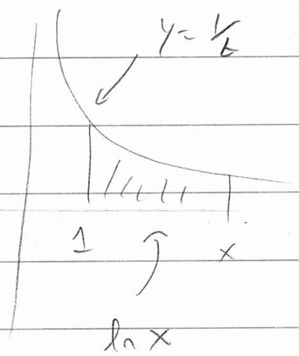


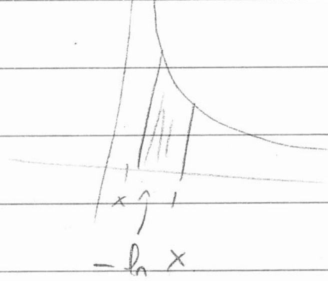

But how fast is it changing?
What is $\frac{d}{dx} \log_a x$??

## The Natural Logarithm

Definition The natural logarithm is defined to be the function

$$\ln x = \begin{cases} \text{the area under the curve } y = 1/t \text{ from } 1 \text{ to } x, & x \geq 1 \\ \text{the negative of the area under the curve } y = 1/t \text{ from } x \text{ to } 1 & 0 < x < 1 \end{cases}$$




Note: domain of $\ln$ is $(0, \infty)$

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**Question:** What is $\lim_{x \to 0^+} \ln x$ ? What is $\ln 1$

What is $\lim_{x \to \infty} \ln x$ ?

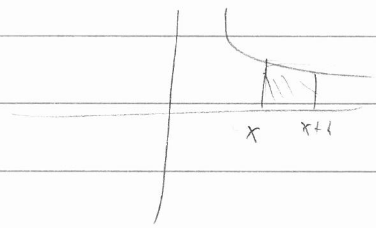
**Theorem:** for $x > 0$ $\frac{d}{dx} \ln x = \frac{1}{x}$

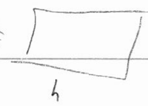
**Proof:** $\frac{d}{dx} \ln x = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$

What is $\ln(x+h) - \ln(x)$? The area under $1/t$ between $x+h$ & $x$

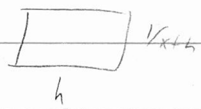
That is

$\ln(x+h) - \ln x =$




So $\ln(x+h) - \ln x \le$ the area of the rectangle  $= \frac{h}{x}$



& $\ln(x+h) - \ln x \ge$ the area of the rectangle  $= \frac{h}{x+h}$



So $\frac{1}{x} = \lim_{h \to 0} \frac{h/(x+h)}{h} \le \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} \le \lim_{h \to 0} \frac{h/x}{h} = \frac{1}{x}$

$\boxtimes$

## Properties of natural logarithm

(1) $\ln(xy) = \ln x + \ln y$

(2) $\ln\left(\frac{1}{x}\right) = -\ln x$

(3) $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln y$

(4) $\ln(x^r) = r \ln x$