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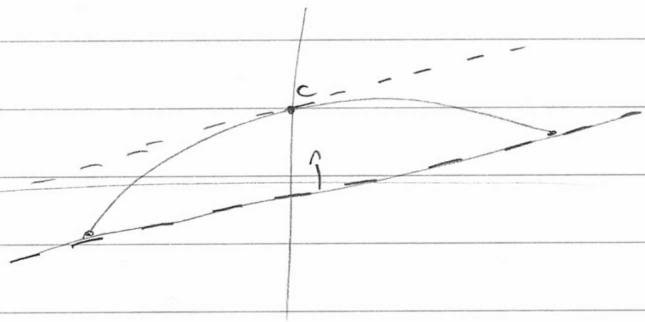
## The mean value theorem

Suppose $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$.
Then there exists a point $c \in (a,b)$ such that

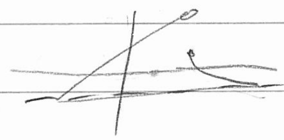
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Graphical 'proof':

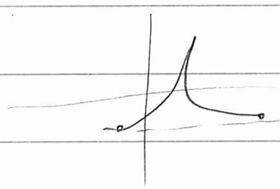



Note: you need all the suppositions

Not continuous on $[a,b]$:



Not differentiable on $(a,b)$:



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Definition Suppose $f$ is defined on an interval, $I$, containing $x_1$ & $x_2$ then if

$f(x_2)$ $>$, $<$, $\geq$, $\leq$ $f(x_1)$ whenever $x_2 > x_1$ then we say $f$ is increasing, decreasing, nondecreasing, nonincreasing on $I$.

Theorem: Suppose $f$ is differentiable on an interval, $I$, then

$f'(x)$ $>$, $<$, $\geq$, $\leq$ $0$ for all $x \in I$, then $f$ is increasing, decreasing, non decreasing, non increasing on $I$

Proof: let $x_1 < x_2 \in I$. By MVT we can find $c \in I$ such that $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$

Then $f(x_2) - f(x_1)$ must have the same sign (or be $0$) as $f'(c)$ and the conclusion holds.

Thm If $f$ is continuous on an interval $I$ and $f'(x) = 0$ for all $x \in I$ then $f(x) = C$, a constant

Proof: By MVT we have $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0$

Thus $f(x_2) = f(x_1)$

<u>Thm</u> If $f$ is defined on an open interval $(a,b)$ and achieves its maximum (or minimum) value at $c \in (a,b)$ then $f'(c)=0$, if it exists.

<u>Proof</u> We have $f(x) - f(c) < 0$

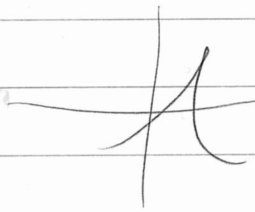
$$0 \leq \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$$

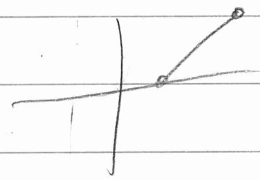
$\boxtimes$

The points at which $f'(c)=0$ are called critical points.

<u>Note:</u> Maxima/minima are not <u>only</u> obtain where the derivative is zero. They can be obtained where the derivative does not exist, or on the endpoints (provided they are in the domain).

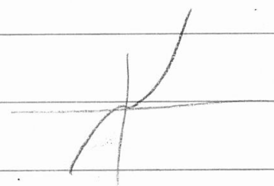
Ex.





Further this does not mean that if $f'(c)=0$ then $c$ is a maxima/minima

<u>Ex:</u> $f(x) = x^3$



Thm Suppose $f$ is continuous on a closed finite interval $[a,b]$ and differentiable on $(a,b)$. If $f(a) = f(b)$ then there exists a $c$ such that $f'(c) = 0$

Proof By Max-min theorem $f$ must have a maxima or minima. This point will have derivative 0.

## Higher order derivatives

We use the notation $f''(x)$ to mean the derivative of the derivative of $f$.

Ex: $f(x) = x^5 \Rightarrow f''(x) = 20x^3$

In general we would write $f^{(n)}(x)$ to mean the $n^{th}$ iterated derivative.

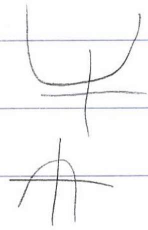
In Liebniz notation we will write

$$\frac{d^n}{dx^n} f(x) \quad \text{or} \quad \frac{d^n y}{dx^n}$$

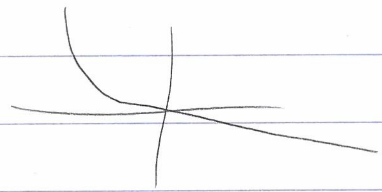
(Think of this as the function $\frac{d}{dx}$ applied $n$ times or $\left(\frac{d}{dx}\right)^n$)

As $f'$ tells us the rate at which $f$ is changing, $f''$ tells us the rate at which $f'$ is changing

So if $f''(x) > 0$ this says that the slope is increasing, which causes a "concave up" shape, while if $f''(x) < 0$ this cause a "concave down" shape



Note: We can have $f'(x) < 0$ while $f''(x) > 0$. This just means that $f$ is decreasing but is slowing



## Implicit Differentiation

Up until now we have only dealt with curves (graphs) that come from functions. i.e. of the form $y = f(x)$. In general curves are of the form
$F(x, y) = 0$ for some expression $F$ involving $x$ and $y$.

For example the circle of radius 5 is given by $x^2 + y^2 - 25 = 0$.

We want to determine how fast $y$ is changing on the circle as $x$ changes (i.e. $\frac{dy}{dx}$)

We can do this by applying the function of functions $\frac{d}{dx}$ to $F(x, y)$ and applying chain rule.

Therefore for $F(x,y) = x^2+y^2-25$

We have $\frac{d}{dx} F(x,y) = \frac{d}{dx}\left(x^2+y^2-25\right)$

$= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(25)$

$= 2x + 2y\frac{dy}{dx} - 0$

Hence on the circle defined by $x^2+y^2-25=0$
We can apply $d/dx$ to both sides to obtain

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Ex: Find $\frac{dy}{dx}$ for $x^3y = 2 - xy^5$

Find the equation of the tangent line at
$2x+y-\sqrt{2}\sin(xy) = \pi/2$ at $\left(\frac{\pi}{4}, 1\right)$

Apply $\frac{d}{dx}$ $\quad 2 + \frac{dy}{dx} - \sqrt{2}\cos(xy)\left(y + \frac{dy}{dx}x\right) = 0$

Plug in $x=\frac{\pi}{4}, y=1 \rightarrow 2 + \frac{dy}{dx} - \sqrt{2}\,\frac{1}{\sqrt{2}}\left(1+\frac{\pi}{4}\frac{dy}{dx}\right) = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-4}{3\pi} \Rightarrow y = \frac{-4}{3\pi}\left(x-\frac{\pi}{4}\right)+1$

$y = \frac{-4}{3\pi}x + \frac{4}{3}$