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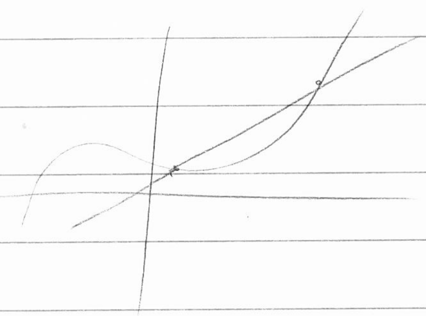
Section 2: The derivative

Recall: Calculus is the study of instantaneous rate of change.

So we want to understand how quickly a function changes over time.

First idea: Secant lines



The secant line between points approximates how quickly the function changes between the two points. i.e. slope of secant line $\approx$ rate of change.

However, we want to know how quickly the function is changing at each point! We can approximate this by taking the secant line at two points near the point we are interested in.

But what two points? If we want to know what is happening near x_0, why not x_0 & x_0+h for h small.

So the rate of change at x_0 is approximated by the slope of the secant line between x_0 & x_0+h.

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But what is the slope of the secant line?
Well, the secant line goes through the points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$. So the slope will be

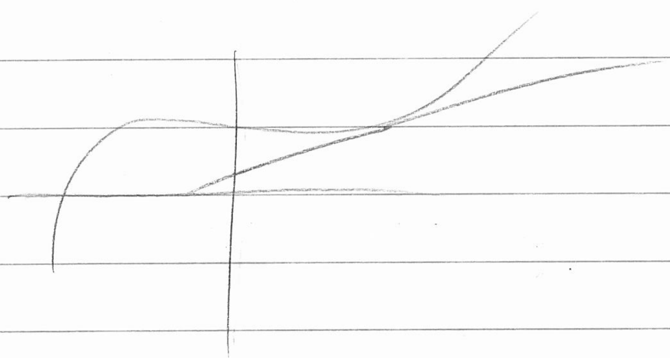
$$\frac{f(x_0+h)-f(x_0)}{x_0+h-x_0} = \frac{f(x_0+h)-f(x_0)}{h}$$

However, we want to know what happens instantaneously at x_0, so we want h to be 0. However, plugging in $h=0$ above gives us "$\frac{0}{0}$" indeterminate. So we want to know what happes as h approaches 0.

Hence we say the rate of change of $f(x)$ at x_0 is

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

If we were to look at what happens to the secant lines as h approaches 0, we would end up with a line that touches our curve at only one point, x_0. This line is called the tangent line of the curve at x_0.



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And insted of rate of change at x_0, we normally say "slope of the tangent line at x_0"

Fact: The tangent line of $f(x)$ at x_0 is the line with slope $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ that passes through the point $(x_0, f(x_0))$

Definition The derivative of a function f is another function, denoted f', defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for all points where the limit exists.

If $f'(x)$ exists then we say f is differentiable at x

So if we write $f: D \longrightarrow R$ then $f': D' \longrightarrow R'$

$x \longmapsto f(x)$ $\qquad$ $x \longmapsto \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Note D' & R' may differ from D & R.

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Using this new notation we may write the tangent line of $f(x)$ at x_0 as:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

<u>Sidenote:</u> If f is defined on a closed interval $[a,b]$, then the definition does not allow us to define the derivative at a or b. So we define the right and left derivative as

$$f_+'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \quad f_-'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

Then we say f is <u>differentiable at a and b</u> if $f_+'(a)$ and $f_-'(b)$ exists.

<u>Definition</u> We say f is <u>differentiable</u> if f' exists for all x in the domain of f

<u>Theorem:</u> f is differentiable implies f is continuous

<u>Proof:</u> If f is differentiable, then $f'(x) = \lim_{h \to 0} (f(x+h) - f(x))/h$ exists for all x. Hence

$$\lim_{h \to 0} (f(x+h) - f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot h = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} h = 0$$

Therefore $\lim_{h \to 0} f(x+h) = f(x)$ ∎

Ex: $f(x) = c$ (constant) $\Rightarrow$ $f'(x) = 0$

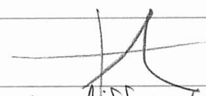
First do $n=2, -1, \frac{1}{2}$ $\nearrow$ $f(x) = x^n$ $\Rightarrow$ $f'(x) = nx^{n-1}$ – power rule.

$f(x) = |x|$ $\Rightarrow$ $f'(x) = \frac{x}{|x|}$

Note here, that $|x|$ is <u>not</u> differentiable at 0.

$f(x) = \frac{x}{|x|}$ $\Rightarrow$ $f'(x) = 0$, $x \neq 0$

Note here that $\lim_{x\to 0} f'(x)$ exists even though neither $f(0)$ nor $f'(0)$ exists.

<u>Also</u>: 
no differentiable.

$f(x)$	$f'(x)$	
$\sin x$	$\cos x$	Prove these
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	these will be important later
$\sec x$	$\sec x \tan x$	
$\cot x$	$-\csc^2 x$	can be shown with quotient rule (later)
$\csc x$	$-\csc x \cot x$	

<u>Recall</u> $\sec x = \frac{1}{\cos x}$ $\csc x = \frac{1}{\sin x}$ $\cot x = \frac{1}{\tan x}$

<u>Important limit</u> $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\left.\begin{aligned} \sin(x+h) &= \sin x \cos h + \cos x \sin h \\ \cos(x+h) &= \cos x \cos h - \sin x \sin h \end{aligned}\right\} \text{p. 51}$$