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Continuity

Definition: If c is in the domain of f (but not an endpoint) then we say f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

If either the limit does not exist or the limit is not equal to c, then we say that f is discontinuous at c

Ex: $\frac{x}{|x|}$ is discontinuous at 0

$$f(x) = \begin{cases} 2x^3 + 7 & x > 5 \\ x^2 - 7x + 10 & x \leq 5 \end{cases}$$ is discontinuous at $x = 5$.

Definition

We say f is left continuous at c if
$\lim_{x \to c^-} f(x) = f(c)$

We say f is right continuous at c if
$\lim_{x \to c^+} f(x) = f(c)$

We say f is continuous at a left endpoint c if it is right continuous

We say f is continuous at a right endpoint c if it is left continuous.

We say f is <u>continuous</u> (on its domain) if f is continuous at every point on its domain.

<u>Note:</u> we can always view f as a function with a smaller domain and so can talk about f being continuous on a subset of the domain i.e. an interval or union of intervals

The graph of a continuous function can be drawn without lifting your pencil

<u>Question:</u> Is $f(x) = \frac{1}{x}$ continuous?

Ans: Yes

Is $f(x) = \sqrt{x}$ continuous?

Ans. Yes.

<u>Examples of continuous functions</u>

i) polynomials
ii) rational functions
iii) rational powers $x^{m/n}$
iv) trig functions (sine, cose, ~)
v) absolute value function.

If f & g are both continuous then so is

i) $f+g$, $f-g$
ii) $f \cdot g$
iii) f/g (on the restricted domain where $g \neq 0$)
iv) $f^{1/n}$ provided $f > 0$ if n is even
v) $f \circ g$ is continuous

These can be proven using the limit properties.

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The max-min theorem

If $f(x)$ is continuous on a closed interval then it obtains its maximum & minimum.

That is, if $f(x)$ is continuous on $[a,b]$ then then exists a $c \in [a,b]$ such that $f(c) \le f(x)$ for all $x \in [a,b]$, and a $d \in [a,b]$ s.t. $f(x) \le f(d)$ for all $x \in [a,b]$.

Note: This is not true if f is not continuous on the interval or if the interval is not closed.

Ex: $f(x) = \frac{1}{x}$ does not achieve its maximum nor it's minimum on the closed interval $[-1,1]$ because it is not continuous at 0.

$f(x) = x^2 - 2x$ does not achieve its maximum on the open interval $(0,3)$ as the interval is open. However it does achieve its minimum

A function may still achieve a maximum or minimum if it is not continuous or on an open interval

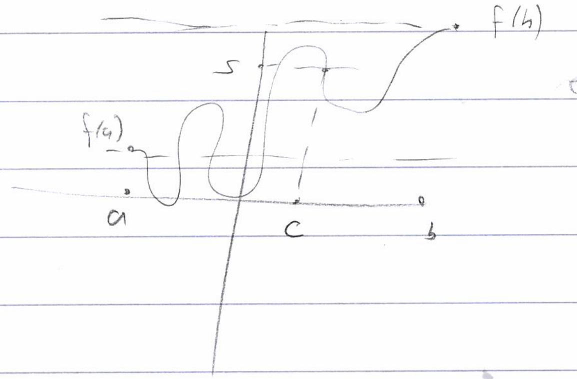
Ex: $\frac{x}{|x|}$ achieves both maximum & minimum on $[-1,1]$ and $\sin x$ achieves both maximum & minimum on $(0, 2\pi)$.

(1a)

Intermediate Value theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and if $f(a) \le s \le f(b)$ then there exist a $c \in [a,b]$ such that $f(c) = s$.

Graphical "proof"



Application: Finding roots of continuous function

If we want to find when a continuous function is 0, then we can find a value for which it is negative and a value for which it is positive and know that it will be zero somewhere between them.

Ex: $f(x) = x^3 - x - 1$

x	$f(x)$
1	-1
2	5
1.5	0.8750
1.25	-0.2969
1.375	0.2246
⋮	
1.3248	0.0003
1.3246	-0.0007

→ Know there is a root between 1.3246 & 1.3248.

Actual root: 1.32471795...

This called the bisection method.