Section 2: Limits

A limit can be thought of as what happens as something gets arbitrarily close to something we may or may not already know about.

Ex: Suppose we want to find the area of a circle of radius 1. We can inscribe polygons with more and more sides inside the circle and calucate their areas. So we would say "the limit of the area of the polygons is the area of the circle"

n	Area of n-gon
4	2
6	2.598
8	2.828
20	3.090
100	3.13953
1000	3.14157.

For a function $f(x)$, we talk about the limit as x approaches some number a. We write this as $\lim_{x \to a} f(x)$.

If $f(x)$ is defined at all values near a, except possibly at a, and if we can show that $f(x)$ gets arbitrarily close to L for all x arbitrarily close to a, the we say

$$\lim_{x \to a} f(x) = L.$$

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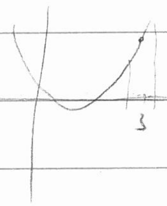
Ex: $f(x) = x^2 - 3x + 2$



$\lim_{x \to 3} f(x) = \lim_{x \to 3} x^2 - 3x + 2 = 2$

x	$f(x)$
2.9	1.71
2.99	1.9701
2.999	1.997
3.1	2.31
3.01	2.0301
3.001	2.003..

In fact we see that $\lim_{x \to 3} f(x) = f(3)$ in this case. $\longrightarrow$ onto back

It is not always the case that $\lim_{x \to a} f(x) = f(a)$ since $f(a)$ may not make sense!

Ex: $f(x) = \frac{x^2 - 4}{x - 2}$ What is $\lim_{x \to 2} f(x)$?

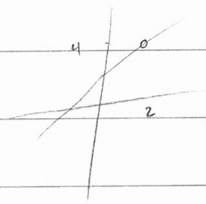
Plugging $x = 2$ in we get "$\frac{0}{0}$", an indeterminant form.

However, if we factor the numerator, we get

$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2$, if $x \neq 2$

i.e. $f(x)$ can be written as a piecewise function:

$$f(x) = \begin{cases} x + 2 & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$$

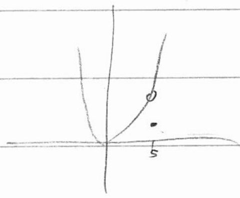


So $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} x + 2 = 4$

Note! This does not mean that the value of $f(x)$ at $x = 2$ is 4!

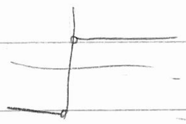
This means that the value of $f(x)$ approaches 4 as x approaches 2!

Ex: $f(x) = \begin{cases} x^2, & x \neq 5 \\ 10, & x = 5 \end{cases}$ What is $\lim\limits_{x \to 5} f(x)$



See by the graph $\lim\limits_{x\to 5} f(x) = 25$
but $f(5) = 10$.

Ex: $f(x) = \frac{x}{|x|}$. What is $\lim\limits_{x \to 0} f(x)$?



We see that $f(x)$ is not defined at 0.
By the graph we see that x approaches 1 if it is positive but -1 if it is negative. Therefore the limit does not exist!

Sometimes write $\lim\limits_{x \to 0} \frac{x}{|x|}$ DNE.

One-sided limits

If $f(x)$ is defined at all values larger than a, except possibly a, and if we can show that $f(x)$ gets arbitrarily close to L for all x arbitrarily close but larger than a, then we say the right limit of $f(x)$ at a is L and we write

$$\lim_{x \to a^+} f(x) = L$$

Similarly if we consider only values smaller than a we get the left limit and write

$$\lim_{x \to a^-} f(x) = L.$$

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Ex: $\lim_{x\to 0^+} \frac{x}{|x|} = 1$ $\lim_{x\to 0^-} \frac{x}{|x|} = -1$

The limit of $f(x)$ at a is L if and only if the right limit and the left limit of $f(x)$ at a are both L:

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L \ \& \ \lim_{x\to a^-} f(x) = L$$

Consequently, if the right limit and the left limit are not equal, we can say that the limit does not exist.

Properties of limits

If $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = M$ then

1. Limit of a sum: $\lim_{x\to a} \left(f(x) + g(x) \right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$
2. Limit of a difference: $\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$
3. Limit of a product: $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L \cdot M$.
4. Limit of a multiple: If b is constant, then $\lim_{x\to a} b f(x) = b \cdot \lim_{x\to a} f(x) = b \cdot L$.
5. Limit of a quotient: If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \left(\lim_{x\to a} f(x)\right) / \lim_{x\to a}(g(x)) = L/M$
6. Limit of a power: $\lim_{x\to a} [f(x)]^{m/n} = \left(\lim_{x\to a} f(x)\right)^{m/n} = L^{m/n}$ provided $L^{m/n}$ exists.
7. Order is preserved: If $f(x) \leq g(x)$ for all x near a then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

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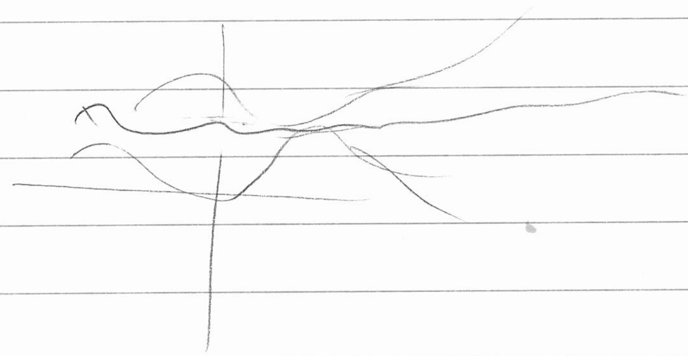
8. The Squeeze Theorem

Suppose $f(x) \le g(x) \le h(x)$ for all x in an open interval containing a, except possibly at a itself.
Suppose also that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

Then $\lim_{x \to a} g(x) = L$.

Graphical "proof"



Limits at infinity

If a function f is defined on the open interval (a, ∞) and if $f(x)$ gets arbitrarily close to L as we take x arbitrarily large then we say that the limit as x approaches ∞ is L and write

$$\lim_{x \to \infty} f(x) = L$$

Likewise if $f(x)$ approaches L as x gets arbitrarily small we say that the limit as x approaches $-\infty$ is L and write $\lim_{x \to -\infty} f(x) = L$

On the graph these are represented as horizontal asymptotes
Note: these limits are necessarily one-sided limits

$\lim_{x\to\infty} \sin x$ DNE $\quad \lim_{x\to\infty} \frac{\sin x}{x}$

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Ex: $\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$

$$\begin{cases} \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{x}{|x|} = 1 \\ \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x}{|x|} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = -1 \end{cases}$$

$$\lim_{x\to\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\infty} \frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x^2}} = \frac{2}{3}$$

$$\lim_{x\to\infty} \frac{5x+2}{2x^3-1} = \lim_{x\to\infty} \frac{5/x^3+2/x^3}{2-1/x^3} = \frac{0}{2} = 0$$

$\lim_{x\to\infty} \sqrt{x^2+x} - x$ — if have time.

Infinite limits

A function that gets arbitrarily large as x approaches a is said to have an infinite limit at a. Since the function does not approach a number, the limit does not exist however we still write

$$\lim_{x\to a} f(x) = \infty.$$

On the graph these are represented as vertical asymptotes.

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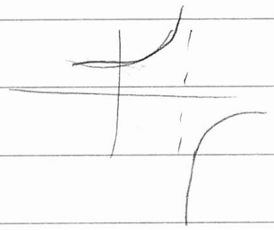
Ex: $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

$\lim_{x\to 2} \frac{x-3}{x^2-4}$ DNE but $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = -\infty$

$\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \infty$

This implies thae graph looks like



In general: if $f(x)$ and $g(x)$ are defined at and near a then

(i) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$ if $g(a) \neq 0$

(ii) $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$ or $-\infty$ if $g(a) = 0$, $f(a) \neq 0$

(iii) $\lim_{x\to a} \frac{f(x)}{g(x)} =$??? if $f(a) = 0$, $g(a) = 0$

Formal definition of a limit (ε-δ definition)

$\lim_{x\to a} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then x is in the domain of f and $|f(x) - L| < \varepsilon$.