1

General overview:

What is Calculus? Calculus is the study of instantaneous rates of change of functions.

For example: If we know the position of an object, as a function of time, can we determine its velocity at a specific time? (derivative)

If we know the velocity as a function of time, can we determine its position? (antiderivative, integral)

Main topics:

1. Functions - how we model physical problems.
2. Limits - how we deal with "instantaneous" phenomenon
3. Derivatives - how we find instantaneous rates of change
4. Complicated Functions - using derivatives to understand complicated functions.
5. Integrals - the "inverse" of derivatives, antiderivatives
6. Applications

Section 1: Functions

Definition: A set is collection of objects (usually numbers)

Ex: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$\mathbb{R}$ = set of real numbers

Intervals $[a, b] = \{x : a \le x \le b\}$ - closed interval

$(a, b) = \{x : a < x < b\}$ - open interval

$(a, b] = \{x : a < x \le b\}$

$[a, b) = \{x : a \le x < b\}$

Set exclusion: $\mathbb{R} \setminus \mathbb{Z}$ = set of real numbers that aren't integers

$\mathbb{R} \setminus \{0\}$ = set of real numbers that aren't 0.

We will use the notation $x \in S$ to mean "x is an element of the set S" and $x \notin S$ to mean "x is not an elements of the set S"

i.e. $1 \in \mathbb{N}$, $-10 \in \mathbb{Z}$, $\pi \in \mathbb{R}$, $-1 \notin \mathbb{N}$, $1/2 \notin \mathbb{Z}$, ..

Definition: A function f is a map from one set D, called the domain, to another set R, called the range.

We often write $f: D \to R$. and if $x \in D$, we write $f(x)$ as what x gets sent to.

Note: $f(x) \in R$.

A Function MUST take every input to only one output

To illustrate this we may also write

$$f : D \to R$$
$$x \longmapsto f(x)$$

Examples $f : \mathbb{R} \to \mathbb{R}$, so $f(x) = x^2$

$x \longmapsto x^2$

And we write $f(1) = 1$ $f(2) = 4$ $f(-\frac{3}{2}) = \frac{9}{4}$ ...

Note that x^2 is always positive, so we could change the range from $\mathbb{R}$ to the positives $[0, \infty)$ to give more information

Some functions have restricted domains, meaning there are some numbers it does not make sense to input. Most notable obstructions are roots and division by zero.

Example: you can't input negative numbers into the function $f(x) = \sqrt{x}$, so the domain is $[0, \infty)$

write $f : [0, \infty) \longrightarrow [0, \infty)$

$x \longmapsto \sqrt{x}$

You can't input zero into the function $f(x) = 1/x$ so the domain is $\mathbb{R} \setminus \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Questions: What is the domain of

$f(x) = \frac{1}{x-2}$, $f(x) = \sqrt[3]{x}$, $f(x) = \frac{x+1}{x^2-9}$

Graphs We sometimes like to view functions graphically. To do this we will frequently write
$y = f(x)$ and then plot the points on the x-y axis

Example: $y = f(x) = x^2 - 2x + 2$

x	$y = f(x)$
-2	$10 = f(-2)$
-1	$5 = f(-1)$
0	$2 = f(0)$
1	$1 = f(1)$
2	$2 = f(2)$

The range of the function will then be all y-values obtained. So the range of the example will be $[1, \infty)$

Note: Not all graphs come from functions.
Since a function must have only one output for every input, a graph that fails the "vertical line test" does not come from a function.

Concrete example: The circle has equation
$x^2 + y^2 = 1$,

Special Functions

1. Linear functions: $y = f(x) = mx + b$

so called because their graphs are lines.

m is called the slope as it is the rate of change of the graph

b is called the y-intercept as it is where the line intercepts the y-axis.

A line passing through the two points (x_1, y_1) & (x_2, y_2) have slope $m = \frac{y_2 - y_1}{x_2 - x_1}$

Ex: Find the equation of the line passing through $(1,3)$ & $(3,4)$ Ans: $y = \frac{1}{2}x + \frac{7}{2}$

Two lines are parallel if they have the same slope!

Ex: $y = 2x+3$ & $y = 2x+10$ & $2y - 4x + 6 = 0$ are all parallel

Two lines are orthogonal if their slopes are negative reciprocals "i.e. minus one over the other"

Ex: $y = 2x + 1$ & $y = -\frac{1}{2}x + 3$ are orthogonal

$2x - 3y = 0$ & $3x + 2y - 2 = 0$ are orthogonal

⑥

(2) Polynomials: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$

Ex: $f(x) = x^2+1$, $f(x) = x^4 + 3x + 2$, $f(x) = 5x^8 + 9x^{14}\ 7x + 6$

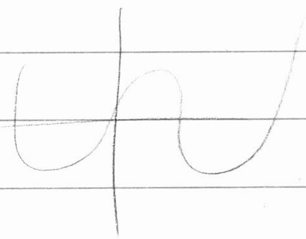
The roots of the polynomial are the values such that $f(x)=0$. Ex: $f(x) = x^3 + 4x^2 - 55x + 50$ has the roots $x = 1, 5, -10$ since $f(1) = f(5) = f(-10) = 0$.

If $f(x)$ has a root r then we can find another polynomial $g(x)$ such that $f(x) = (x-r)g(x)$

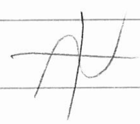
Ex. $x^3 + 4x^2 - 55x + 50 = (x-1)(x^2+5x-50) = (x-1)(x+10)(x-5)$

A polynomial will have at most as many zeroes as the highest power of x appearing. Generically, it will have exactly that many

Graph of a generic quartic



Graph of a generic cubic



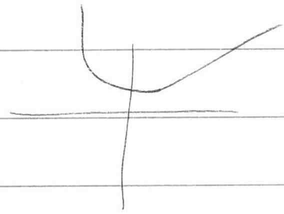
(3) Rational functions: $f(x) = \frac{P(x)}{Q(x)}$ where P & Q are polynomials. They are not defined at the roots of

$f(x) = \dfrac{x+1}{x^2-4}$ $\quad f(x) = \dfrac{x^2-1}{x^2+2}$

Domains?

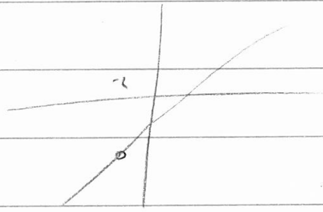
(4) Piecewise defined functions

Ex: $f(x) = \begin{cases} x^2+1 & x \le 1 \\ 2x & x > 1 \end{cases}$



Ex $f(x) = \dfrac{x^2-4}{x+2} = \dfrac{(x+2)(x-2)}{(x+2)} = x-2, \quad x \neq -2$

$f(x) = \begin{cases} x-2 & x \neq -2 \\ \text{undefined} & x = -2 \end{cases}$



Absolute value function: $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

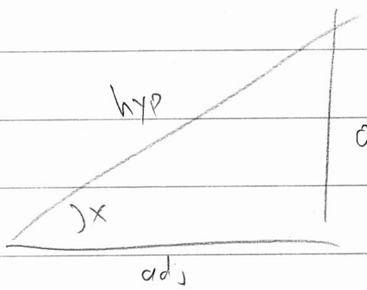
i.e. leaves x alone if positive, but makes x positive if x is negative.

Ex: $f(1) = |1| = 1 \qquad f(-2) = |-2| = 2$

Ex: $\dfrac{x}{|x|} = \begin{cases} \dfrac{x}{x} & \text{if } x > 0 \\ \dfrac{x}{-x} & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \end{cases} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0. \end{cases}$

(5) Trig functions

Given a right angle triangle with one angle x

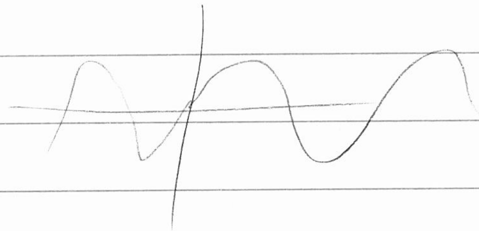


We define

$\sin(x) = \dfrac{\text{opp}}{\text{hyp}} \qquad \cos(x) = \dfrac{\text{adj}}{\text{hyp}} \qquad \tan(x) = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sin(x)}{\cos(x)}$

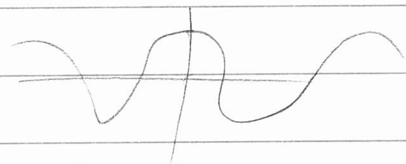
(8)

Graph for $\sin(x)$



Explain how.

Graph for $\cos(x)$



Several useful identities on pages 48 – 52. Worth having a look at. We'll come back to them when they are more relevant (integration)

Most important ones:

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \qquad \sin^2 x = \frac{1 - \cos(2x)}{2}$$