7. Graph Theory

Graph theory is an enormous area that includes problems of different kinds: combinatorial problems, opimisation problems, modelling problems, etc.

In this chapter we'll cover some basic problems and methods in graph theory, with connections to earlier chapters (for instance Stirling's formula and optimisations problems in chapter 4, generating functions in chapter 6) as well as to later chapters (graphs as networks in chapter 8, matching in chapter 9).

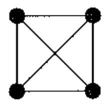
Highlights from this chapter

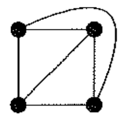
- A graph is said to be *planar* if it can be drawn on a plane without any edges crossing each other. Then the plane is subdivided into a number of regions that are called *faces.*
- *Euler's formula* gives the following relationship between the number of nodes (v), edges (e), and faces (f) in every connected planar graph: $v - e + f = 2$.
- A graph is *homeomorphic* to another one if they look the same when you ignore the nodes: ●——●——● is thus homeomorphic to ●————●. *Kuratowski's theorem* says that a graph is planar if and only if no subgraph is homeomorphic to K_5 or $K_{3,3}$.
- An *proper colouring* of a graph is an assignement of colours to the nodes so that adjacent nodes don't have the same colour. The least number of colours needed for a proper colouring of G is the *chromatic number* $\chi(G)$. The number of possible proper colourings of G if λ colours are available is the *chromatic polynomial* $P_G(\lambda)$.
- The famous *four-colour theorem* states that every planar graph can be coloured using at most four colours.
- The numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$ are called *Catalan numbers.* The Catalan numbers count many different combinatorial objects, among others the number of binary trees with n nodes.
- *Cayley's theorem* states that the number of trees with n differently labeled nodes is given by the simple formula n^{n-2}. This we prove using the *Prüfer code*, a bijection between trees of this kind and $(n-2)$-tuples of numbers between 1 and n.
- A *random walk* on a graph is a walk that in each node randomly chooses which edge it will follow in the next step.

7.1 Planar Graphs

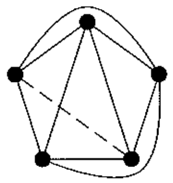
A manufacturer of integrated circuits make up the circuits one layer of conductive connectors at a time. Each layer can be regarded as a graph where the connectors make up the edges and the connecting points the nodes. To avoid short circuits, no connectors may cross each other in the same layer. It must be possible to model the layers using graphs that are **planar**, that is, that can be drawn in the plane without any edges crossing each other.

Example 7.1 The complete graph with four nodes, K_4, is planar.



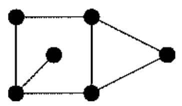


The complete graph with five nodes, K_5, is on the other hand not planar. No matter how you try to draw it some edges will cross – try it yourself!



A formal proof showing that K_5 isn't planar will be given below. ■

A planar graph drawn in the plane without any intersecting edges will generate a partitioning of the plane in **regions**, which are called the **faces** of the graph. For instance, K_4 divides the plane into four regions: three inner ones and one outer one. So the unlimited outer region is counted as a face as well. The graph below thus has three faces: the square one, the triangular one, and the area outside.



Note that every limited face is surrounded by edges that form a cycle.

It's not obvious that every way of drawing the graph will result in the same number of faces, but this is the case. That follows from Euler's formula from 1752:

Theorem 7.1: ***Euler's formula*** In every connected planar graph the following holds:

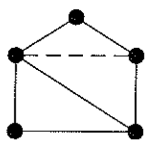
$$v - e + f = 2$$

where v is the number of nodes, e is the number of edges, and f is the number of faces.

Proof. We make an induction over the number of faces. A planar connected graph with a single face (the unlimited one) has to be a tree, since it's a connected graph without cycles. For all trees, we know that $v = e + 1$, and since $f = 1$ it satisfies the identity $v - e + f = 2$.

Now assume that for some $f' \geq 1$ the identity is true for all connected planar graphs with f' faces. We are to prove that it then is true for all connected planar graphs with $f = f' + 1 \geq 2$ faces. Let G be such a graph with e edges,

v nodes, and f faces. G will have some limited face. If we remove an edge from the surrounding cycle, this face will meld into the face on the other side of the edge. Then we have a new planar graph with $e' = e - 1$ edges and $f' = f - 1$ faces (and $v' = v$ nodes).

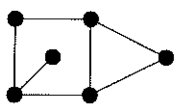


In this new graph the identity, according to the inductive hypothesis, $v' - e' + f' = 2$, is true. Thereby, the same thing holds for the original graph:

$$v - e + f = v' - (e' + 1) + (f' + 1) = v' - e' + f' = 2.$$

According to the principle of induction, the identity thus is true for all planar connected graphs. ∎

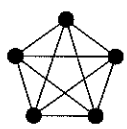
As an example, we can take another look at the graph



This graph has six nodes, seven edges, and three faces; and indeed $6 - 7 + 3 = 2$.

With the use of Euler's formula it can be proved that some graphs can't possibly be planar, namely if it's impossible for them to satisfy the relationship between the numbers of nodes, edges, and faces. But face is a concept only defined in planar graphs. Euler's formula is thus not immediately applicable until you know whether the graph is planar. But every face in a planar graph (that isn't a tree) is surrounded by cycles, and since cycle is a well defined concept in all kinds of graphs we can use them when calculating.

Example 7.2 We are to prove that K_5 isn't a planar graph.



In a planar graph, each face is surrounded by a cycle. All cycles in K_5 of course consist of at least three edges. (Besides, a planar graph may include extra edges that point into the faces.) On the other hand, each edge in a cycle surrounding a face is bordering to exactly two faces. If there is a large enough number of edges to surround all the faces in a planar graph, then the inequality $2e \geq 3f$ has to hold, or if you prefer, $f \leq \frac{2}{3}e$. According to Euler's formula it would then hold, if K_5 were planar, that

$$2 = v - e + f \leq v - e + \frac{2}{3}e = v - \frac{1}{3}e.$$

But in K_5 $v = 5$ and $e = \binom{5}{2} = 10$ and it's not true that $2 \leq 5 - \frac{1}{3} \cdot 10$. So K_5 can't be planar. ∎

In the above example we have proved a corollary of Euler's formula:

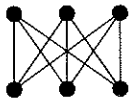
Corollary 7.2 In every planar graph with v nodes and $e \geq 2$ edges it holds that

$$e \leq 3v - 6.$$

(Is derived from Euler's formula, which includes the condition "connected", and is based on the fact that each face has at least three edges, which doesn't hold in K_5.) ■

The number of edges in a planar graph is thus limitied by a linear function in the number of nodes. For arbitrary graphs the number of edges can grow quadratically compared to the number of nodes; K_n has $\binom{n}{2} = \frac{n^2-n}{2}$ edges. That means that the proportion of planar graphs decreases the larger the number of nodes gets.

Example 7.3 We will now prove that the complete bipartite graph $K_{3,3}$ isn't a planar graph.



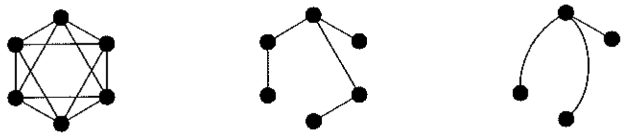
The graph $K_{3,3}$ has $v = 6$ and $e = 3 \cdot 3 = 9$ and thus satisfies the inequality $e \leq 3v - 6$, which according to the corollary above has to hold in all planar graphs. But we can't thereby draw the conclusion that $K_{3,3}$ is planar, since there may exist nonplanar graphs that satisfies the inequality as well.

Instead we are going to sharpen the inequality. Since $K_{3,3}$ is a bipartite graph, all cycles have an even length. All cycles in $K_{3,3}$ thus consist of at least four edges. Thus, according to the same line of reasoning as above, it holds that if $K_{3,3}$ were planar the relationship between faces and edges would satisfy $f \leq \frac{2}{4}e$. According to Euler's formula it then holds, if $K_{3,3}$ were a planar graph, that

$$2 = v - e + f \leq v - e + \frac{2}{4}e = v - \frac{1}{2}e.$$

We have $v = 6$ and $e = 9$, and it isn't true that $2 \leq 6 - \frac{1}{2} \cdot 9$. Thus $K_{3,3}$ can't be planar. ■

We are now going to explain why the two graphs K_5 and $K_{3,3}$ are the most important nonplanar graphs. By a **minor** of a graph G is meant a graph that can be created by performing the following operations on G: It's permitted to remove edges and nodes and for nodes with just two edges it's permitted to fuse these edges into *one* edge (so that the node no longer is a node but is included as one point out of many on the edge).



It's clear that if a graph is planar from start it's still planar if you remove a node or an edge, as well as if you fuse two edges as described above; no intersecting edges are formed by these operations. Thereby we have realised that every minor of a planar graph is planar as well.

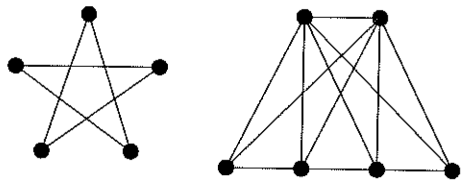
Since nonplanar graphs exist there must exist some *minimal* nonplanar graphs as well, that is, graphs that aren't planar while all their minors are planar. The Polish mathematician Kazimierz Kuratowski (1896–1980) proved in 1930 that there only exist two minimal nonplanar graphs and these are K_5 and $K_{3,3}$. All others include one of these as a minor.

Theorem 7.3: ***Kuratowski's Theorem*** A graph is nonplanar if and only if it includess K_5 or $K_{3,3}$ as a minor. ■

The proof in one direction is in principle already done; a planar graph has only planar minors but K_5 and $K_{3,3}$ are nonplanar. Therefor every graph having K_5 or $K_{3,3}$ as a minor has to be nonplanar.

In the other direction the proof is complicated and won't be given here. Instead, we refer to the book *Graphs on Surfaces* (2001) by the distinguished graph theorists Bojan Mohar and Carsten Thomassen. This book contains algorithms that in linear time determine whether a graph is planar as well.

Exercise 7.1 Show that the following three graphs are planar by drawing them without crossing edges.



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Exercise 7.2 Count the number of faces of each of the graphs above and verify that Euler's formula holds in all of them. *

Exercise 7.3: ***Important!*** Give examples of more situation besides manufacturing of cicuits where planar graphs are useful in modelling.

Exercise 7.4: ***Important!*** Two edges that meet at a node of degree two look like a single edge (since you can imagine the node being a point, that is, without length and width):

The symbol $\simeq$ denotes the equivalence relation **homeomorphism** between the graps. Two graphs are **homeomorphic** if they only differ in nodes of degree two in this way. Prove that if two graphs are homeomorphic then either both are planar or both nonplanar.

Exercise 7.5: ***Important!*** Express Kuratowski's theorem using the concept homeomorphism instead of the concept minor.

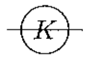
Exercise 7.6: ***Important!*** Why is a graph nonplanar if some subgraph is nonplanar? Give examples showing that the converse isn't true.

Exercise 7.7 Study a planar connected graph with at least one cycle. If all the cycles are of length at least $k \geq 3$, derive the inequality

$$e \leq \frac{k}{k-2}(v-2).$$

Exercise 7.8 Show that the problem of determining whether a graph is nonplanar belongs to NP.

Exercise 7.9 Kimmo once got an idea about how to draw even nonplanar graphs without intersecting edges. His innovation was to introduce two new kinds of nodes, "K_n-nodes" and "$K_{m,n}$-nodes", that are drawn like

 and 

These new kinds of nodes are defined so that all nodes in the graphs that are adjacent to a certain K_n-node are counted as adjacent to each other, while all nodes in the graph that are adjacent of one side of a $K_{m,n}$-node by this are adjacent to all the nodes in the graph that are adjacent to the other side of the $K_{m,n}$-node.

The idea is thus that a number of edges in the original graph are replaced by such a modern node and edges to it. For instance the graph

means that the three normal nodes are adjacent to each other, while the graph

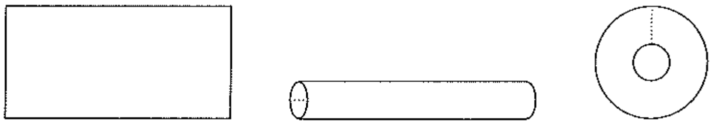
means that the two nodes to the left are adjacent to the node to the right but not to each other.

(a) Show how it's possible using K_n-nodes and $K_{m,n}$-nodes to draw K_5 and $K_{3,3}$ without any intersecting edges.

(b) Kimmo thought that the exercise above combined with Kuratowski's theorem ought to show that *all* graphs could be drawn without intersecting edges by using K_n-nodes and $K_{m,n}$-nodes. Explain to Kimmo why this isn't correct!

Exercise 7.10: ***Important!*** There exist other surfaces than the plane on which to draw graphs. For instance you can draw on the outside of a sphere (the outside of a beach ball) or on a torus (the outside of a swimming ring). If you reflect for a while, you realise that it's just as difficult to draw a graph without intersecting edges on a sphere as on the paper. If you draw a very small graph, the part of the sphere where you are drawing can be regarded as flat, so everything that can be drawn without intersecting edges on paper can be drawn on a sphere. And conversely; if you have drawn on a sphere (say a balloon) you can cut a small hole in one face and stretch the sphere like a drumhead. No edge crossings are formed by that operation, so anything that can be drawn without intersecting edges on a sphere can be draw on a plane as well.

But it's easier to draw without intersecting edges on a torus than it is on a flat paper! A torus can be manufactured by taking a paper, turning it into a roll and then gluing the ends together:



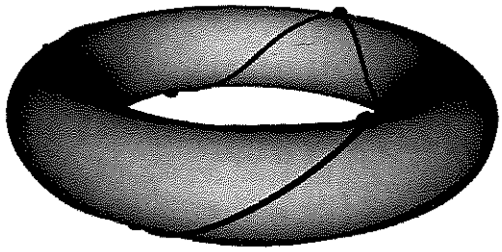


Figure 7.1: *Torodial graph.*

When you are drawing on a paper representing a tours you are thus allowed to draw up to the upper edge and then continue from the corresponding point at the lower edge, or to draw out to the right edge and then continue from the corresponding point at the left edge, since these points are glued together. Then we can in fact draw both K_5 and $K_{3,3}$ without intersecting edges. Let's say that a graph is **torodial** if it can be drawn on a torus without intersecting edges.

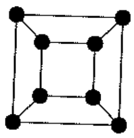
(a) Show that K_5 is torodial.

(b) Show that $K_{3,3}$ is torodial.

(c) Why doesn't the reasoning in Euler's formula hold when we are drawing on a torus instead?

On a torus, other graphs than K_5 and $K_{3,3}$ are the minimal graphs that can't be drawn without intersecting edges, and they are a considerably larger number than two. It's still not known what all of them are, but at least it's known that the number is finite. Read more in the book by Mohar and Thomassen.

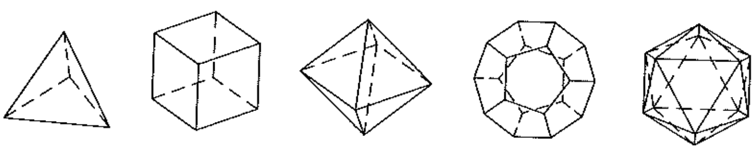
Exercise 7.11 A convex **poyhedron** is a convex three-dimensional solid with flat faces, for instance a cube. The faces are in their turn circumscribed by edges and corners and thus form a graph. For example, the graph of the cube looks like



Explain why the graph of every polyhedron is planar.

Exercise 7.12 Do all planar graphs correspond to polyhedrons?

Exercise 7.13 A **Platonian solid** is a polyhedron that is completely regular. Use Euler's formula to prove that the only Platonian solids are the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. Draw the solids in a planar way as well.

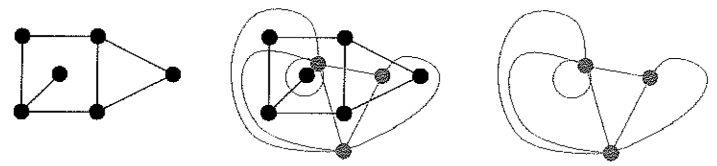


7.1.1 Dual Graphs

For each planar graph G that is drawn without intersecting edges a **dual graph** $G^{\perp}$ is defined as

- each region in G gives a node in the dual graph $G^{\perp}$
- each edge in G gives an edge in the dual graph $G^{\perp}$, so that adjacent regions in G correspond to adjacent nodes in $G^{\perp}$.

Example 7.4 The figure shows a planarly drawn graph and how we based on the drawing get the dual graph. (Note that a simple graph can have a multigraph as its dual!)



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Exercise 7.14 Can different ways of drawing a planar graph G give different nonisomorphic dual graphs?

Exercise 7.15 Find the dual graphs of the planar graphs

(a) K_2

(b) K_3

(c) K_4

(d) $K_{2,2}$

(e) $K_{2,3}$ *

Exercise 7.16 Find the dual graphs of the planar graphs

(a) Tetrahedron (see exercise 7.13).

(b) Cube.

(c) Octahedron.

(d) Dodecahedron.

(e) Icosahedron.

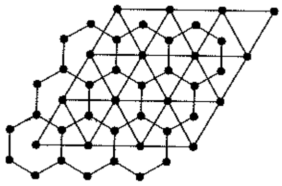
Exercise 7.17 How can a world map be represented by a graph where the countries correspond to nodes?

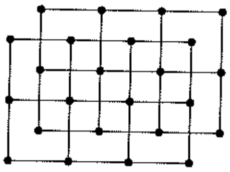
Exercise 7.18 Explain why the dual graph of a planar graph has to be planar as well. *

Exercise 7.19 What is the dual of the dual $(G^{\perp})^{\perp}$?

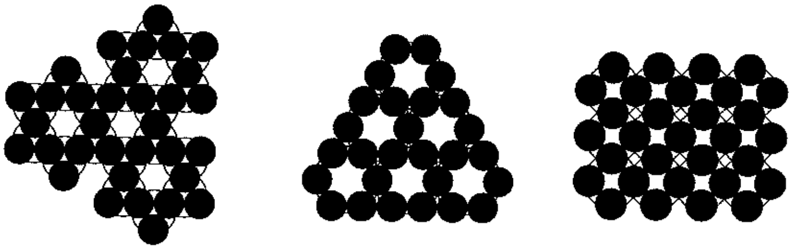
It's of special interest to study the dual graphs of the Platonian solids. You will find that the tetrahedron is its own dual, while cube–octahedron and dodecahedron–icosahedron are dual pairs. It's thus possible to inscribe a octahedron into a cube, that in its turn is inscribed into an octahedron.... A consequence of this is that you can make up hybrids of the solids. The third graph in exercise 7.1 is a **cube-octahedron**. You can either regard it as a cube where the eight corners have been cut off, resulting in eight triangular surfaces, *or* as an octahedron where the six corners have been cut off, resulting in six square surfaces.

Similar things are of interest in tiling. There are three ways of tiling a floor using regular polygons: you can use triangles, hexagons, or squares. Triangles and hexagons are the duals of each other while the square tiling is the dual of itself.





This in its turn has interesting applications for persons who make bracelets out of beads. From the triangular tiling you get a structure where loops consisting of three beads are combined in groups of six; from the hexagonal tiling one where loops consisting of six beads are combined in groups of three. And from the square tiling one where loops of four beads are combined in groups of four.



That the square tiling is the dual of itself gives a form of double symmetry in the design, since the "loops" and the "crossings" are equal. That in its turn has interesting consequences when designing the pattern.

Exercise 7.20 If you look att the bead constructions drawn here, you see that they can be regarded as graphs, with the beads as nodes and the thread as edges. Then you find that all the nodes have an even degree. What is the consequence of this?

Exercise 7.21 Of course three-dimensional graphs can be built out of beads as well. Buy a bag of beads and make the Platonian solids! (Hint: It's easier if you put needles at both ends of the thread.) *

7.2 Graph Colouring

An optimisation problem, which is important in radio communication, is the **frequency assignment problem**: A number of radio towers are placed in an area. Some towers are so close that they would interfere with each other if they were sending on the same frequency. Other towers are so far apart that they can send on the same frequency without problems. How many frequencies are needed if one wants to use as few frequencies as possible?