SF1684 Algebra and Geometry

Lecture 2

Matrix operations, inverse matrices, subspaces

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Outline

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Matrices

Matrix

A general $m \times n$ matrix A has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}.$$

Alternative notations for the entry a_{ij} are $(A)_{ij}$ or $A[i,j]$.

Square matrix

If $n = m$ then A is called a **square matrix**. The **diagonal entries** of A are $a_{11}, a_{22}, \ldots, a_{nn}$.

Row and column vectors

We may represent A as the list of n column vectors or m row vectors

$$A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$$

where

$$\mathbf{c}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m \quad \text{and} \quad \mathbf{r}_i(A) = \begin{bmatrix} a_{i1} \; a_{i2} \; \cdots \; a_{in} \end{bmatrix} \in \mathbb{R}^n.$$

Matrices with special forms

Let A be a square matrix.

Diagonal matrix

A is called a **diagonal matrix** if all entries outside the main diagonal are zero. In addition, if the diagonal entries are all 1 then A is called an **identity matrix**.

Triangular matrix

A is called an **upper triangular** if all entries below the main diagonal are zero and called a **lower triangular** if all entries above the main diagonal are zero.

Symmetric and skew-symmetric matrix

A is called **symmetric** if all entries are symmetric with respect to the main diagonal, that is $(A)_{ij} = (A)_{ji}$, and called **skew-symmetric** if $(A)_{ij} = -(A)_{ji}$.

Operations on matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices.

Equal matrices

$A = B$ if and only if they have the same size and $a_{ij} = b_{ij}$ for all i, j.

Matrix addition and subtraction

If A and B have the same size then

$$(A+B)_{ij} = a_{ij} + b_{ij},$$
$$(A-B)_{ij} = a_{ij} - b_{ij}.$$

Scalar multiplication

If $c \in \mathbb{R}$ then

$$(cA)_{ij} = ca_{ij}.$$

Matrix multiplication

If A is an $m \times s$ matrix and B is an $s \times n$ matrix, then the product AB is the $m \times n$ matrix defined as

$$(AB)_{ij} = \mathbf{r}_i(A) \cdot \mathbf{c}_j(B) = \sum_{k=1}^{s} a_{ik}\, b_{kj}.$$

Matrix power

If A is a square matrix, then we define the nonnegative integer powers of A to be

$A^0 = I$ and $A^m = AA \cdots A$ (matrix product of m copies of A).

Matrix polynomial

If A is a square matrix, then the matrix polynomial in A has form

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m.$$

Matrix transpose

If A is an $m \times n$ matrix then the transpose of A, denoted by A^T, is the $n \times m$ matrix defined as

$$(A^T)_{ij} = a_{ji}.$$

Trace

If A is a square matrix of order n, that is $A = [a_{ij}]_{n \times n}$, then the trace of A is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^m$ be two column vectors.

Matrix inner product

If $\mathbf{u}$ and $\mathbf{v}$ have the same size, that is $n = m$, then

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i = \mathbf{u} \cdot \mathbf{v} \in \mathbb{R}.$$

Matrix outer product

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} = [u_i v_j]_{n \times m} \in \mathbb{R}^{n \times m}.$$

Matrix products as linear combinations

If A is an $m \times s$ matrix, B is an $s \times n$ matrix, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^s$ are two column vectors then

$$A\mathbf{u} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_s \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_s \end{bmatrix} = u_1\mathbf{c}_1(A) + u_2\mathbf{c}_2(A) + \cdots + u_s\mathbf{c}_s(A)$$

which is a linear combination of the column vectors of A with coefficients from $\mathbf{u}$. Similarly

$$\mathbf{v}^T B = \begin{bmatrix} v_1 & v_2 & \cdots & v_s \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_s \end{bmatrix} = v_1\mathbf{r}_1(B) + v_2\mathbf{r}_2(B) + \cdots + v_s\mathbf{r}_s(B)$$

which is a linear combination of the row vectors of B with coefficients from $\mathbf{v}$.

Theorem

Columns and rows of the product AB as linear combinations

$$AB = \begin{bmatrix} A\mathbf{c}_1(B) & A\mathbf{c}_2(B) & \cdots & A\mathbf{c}_n(B) \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1(A)B \\ \mathbf{r}_2(A)B \\ \vdots \\ \mathbf{r}_m(A)B \end{bmatrix}.$$

Algebraic properties

If A, B, and C are matrices and a, b are scalars, then

Addition and scalar multiplication

- i $A + B = B + A$ [Commutative law for addition]
- ii $A + (B + C) = (A + B) + C$ [Associative law for addition]
- iii $(ab)A = a(bA)$
- iv $(a \pm b)A = aA \pm bA$
- v $a(A \pm B) = aA \pm aB$

Matrix multiplication

- i $A(BC) = (AB)C$ [Associative law for multiplication]
- ii $A(B \pm C) = AB \pm AC$ [Left distributive law]
- iii $(B \pm C)A = BA \pm CA$ [Right distributive law]
- iv $a(AB) = (aA)B = A(aB)$

Warning

In general, the **commutative law** and **cancellation law** do not hold for matrix multiplication.

Example

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}.$$

- $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$.
- $AB = AC = 0$ but all $A, B, C \neq 0$ and $B \neq C$.

Remark

Powers of a square matrix A commute, that is $A^m A^n = A^n A^m = A^{n+m}$ for $n, m \in \mathbb{N} \cup \{0\}$.

Properties of the transpose

i. $(A^T)^T = A$
ii. $(A \pm B)^T = A^T \pm B^T$
iii. $(cA)^T = cA^T$
iv. $(AB)^T = B^T A^T$

Transpose and dot product

If $A \in \mathbb{R}^{m\times n}$ and $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$ are column vectors then

v. $(A\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (A^T\mathbf{v})$

Properties of the trace

If A and B are square matrices with the same size then

i $\operatorname{tr}(A^T) = \operatorname{tr}(A)$

ii $\operatorname{tr}(A \pm B) = \operatorname{tr}(A) \pm \operatorname{tr}(B)$

iii $\operatorname{tr}(cA) = c \operatorname{tr}(A)$

Trace of matrix product

If A is an $m \times n$ matrix and B is an $n \times m$ matrix then

iv $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

In particular, if $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ column vectors then

v $\operatorname{tr}(\mathbf{u}\mathbf{v}^T) = \mathbf{v}^T\mathbf{u} = \mathbf{u} \cdot \mathbf{v}$

Outline

Identity matrix

Definition

The identity matrix of size n, denoted I_n, is the $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere.

Important property: If A is any $m \times n$ matrix, then

$$AI_n = A \quad \text{and} \quad I_m A = A.$$

Echelon form of a square matrix

If R is the reduced row echelon form of an $n \times n$ matrix, then either R has a zero row or R is the identity matrix I_n.

Inverse of a matrix

Definition

If A is an $n \times n$ is square matrix, an inverse of A is an $n \times n$ matrix B with the property that

$$AB = BA = I_n.$$

If such an B exists, then A is called **invertible** (or **nonsingular**). Otherwise, A is called **singular**.

Uniqueness of inverse

If A is an invertible matrix, then its inverse is unique. We will denote the inverse by A^{-1}.

Properties of inverses

If A and B are invertible matrices with the same size then

i. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
ii. kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$ for $k \in \mathbb{R}$, $k \neq 0$.
iii. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
iv. A^m is invertible and $(A^m)^{-1} = (A^{-1})^m = A^{-m}$ for $m \in \mathbb{N}$.
v. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Inverse of a 2×2 matrix

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\det(A) = ad - bc \neq 0$, and the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Elementary matrices

Our goal is to find matrices that emulate the elementary row operations on a matrix. For any $m \times n$ matrix A, and a matrix B equal to A after a row operation, we wish to find a matrix E of size $m \times m$ such that $B = EA$.

$\rightarrow E$ is called the **elementary matrix**, and can be obtained by performing the same row operation on I_m.

Invertibility of elementary matrices

Each elementary matrix E is invertible, and its inverse is an elementary matrix results by performing the inverse of the elementary row operation which produced E from I.

Finding inverses

If A is a $n \times n$ matrix and R is denoted the reduced row echelon form of A then there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = R.$$

1. If R is the identity matrix I_n then $A = E_1^{-1} E_2^{-2} \cdots E_k^{-1}$. Also A is invertible and $A^{-1} = (E_1^{-1} E_2^{-2} \cdots E_k^{-1})^{-1} = E_k \cdots E_2 E_1$.
2. If R is not I_n (R has a row of zeros), then A is not invertible and cannot be represented as a product of elementary matrices.

Theorem

If a sequence of elementary row operations reduces a square matrix A to I, then the same sequence of elementary row operations transforms I into A^{-1}.

Gauss-Jordan method for finding inverse matrix

1. Form $n \times 2n$ matrix with left half A and right half I_n.
2. Apply Gauss-Jordan elimination to the whole matrix (to rows of length $2n$) until the matrix A (in the left half) is in reduced row echelon form.
3. If the left half is I_n then A is invertible and the right half is A^{-1}. Otherwise (the left half is not I_n), A is not invertible (and the right half is of no use).

Example

Find the inverse of $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \mapsto (-1)R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 + (-2)R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 3 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \mapsto (1/3)R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 2/3 & 1/3 \end{array}\right].$$ Thus, $A^{-1} = \begin{bmatrix} -1 & 0 \\ 2/3 & 1/3 \end{bmatrix}$.

Fundamental theorem of invertible matrices

If A is an $n \times n$ matrix, then the following statements are equivalent

a. The reduced echelon form of A is I_n.
b. A is a product of elementary matrices.
c. A is invertible.

Theorem

Let A and B be square matrices of the same size

i. If $AB = I$ or $BA = I$ then A and B are both invertible and each is the inverse of the other.
ii. If the product AB is invertible then A and B are both invertible.

Row equivalent

Two matrices that can be obtained from one another by finite sequences of elementary row operations are said to be **row equivalent**.

Theorem

If A and B are square matrices of the same size, then the following statements are equivalent

a. A and B are row equivalent.
b. There is an invertible matrix E such that $B = EA$.
c. There is an invertible matrix F such that $A = FB$.

Remark

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n.

Solving linear systems using matrices

Matrix equation

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \longleftrightarrow A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Theorem

If $A\mathbf{x} = \mathbf{b}$ is a linear system of n equations in n unknowns then the system has exactly one solution if and only if A is invertible, in which case the solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$.

Fundamental theorem of invertible matrices (cont.)

If A is an $n \times n$ matrix, then the following statements are equivalent

a. The reduced echelon form of A is I_n.
b. A is a product of elementary matrices.
c. A is invertible.
d. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.
f. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.

Outline

Subspaces of $\mathbb{R}^n$

Subspace

Let S be a collection vectors in $\mathbb{R}^n$. Then S is called a subspace of $\mathbb{R}^n$ if

1. The zero vector $\mathbf{0}$ is in S.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in S, then $\mathbf{u} + \mathbf{v}$ is in S (S is closed under addition).
3. If $\mathbf{u}$ is in S and c is a scalar, then $c\mathbf{u}$ is in S (S is closed under scalar multiplication).

Linear combinations

If $\mathbf{v}_1, \dots, \mathbf{v}_k$ are vectors in $\mathbb{R}^n$, and S is the set of all linear combinations of these vectors

$$S = \{t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k \mid t_1, \dots, t_k \in \mathbb{R}\},$$

then S is a subspace of $\mathbb{R}^n$.

Span and spanning set

The subspace S is called the span of $B = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, and we also say that B spans S. The set B is called a spanning set for the subspace S. We denote S by

$$S = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}.$$

Example

If $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the standard unit vectors in $\mathbb{R}^n$ then

$$\mathbb{R}^n = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}.$$

Theorem

Every subspace of $\mathbb{R}^n$ is the span of at most n vectors.

Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

The zero subspace	The zero subspace
Lines through $\mathbf{0}$	Lines through $\mathbf{0}$
All of $\mathbb{R}^2$	Planes through $\mathbf{0}$
	All of $\mathbb{R}^3$

Linear independence

Linear independence

A set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is said to be linearly independent if the only solution to the equation

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

is $t_1 = \cdots = t_k = 0$; otherwise, S is said to be linearly dependent.

Linear dependence

A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ with $k \geq 2$ in $\mathbb{R}^n$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors in S.

Solution space of a homogeneous linear system

Theorem

If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system with n unknowns, then its solution set is a subspace of $\mathbb{R}^n$.

The solution space must be expressible in the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k$$

which we call a **general solution** of the system. This expression is not unique. However the Gauss-Jordan elimination always produce the same general solution and the spanning vectors $\mathbf{v}_i$ in this case are **linearly independent**. The number of vectors (or the number of parameters t_i) is call the **dimension** of the solution space.

Equal matrices

Let A and B be $m \times n$ matrices

- i The solution space of $A\mathbf{x} = \mathbf{0}$ is all of $\mathbb{R}^n$ if and only if $A = 0$.
- ii $A = B$ if and only if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Relationship between $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

Translated solution space

If W is the solution space of $A\mathbf{x} = \mathbf{0}$ then the solution set of the consistent linear system $A\mathbf{x} = \mathbf{b}$ is the translated subspace $\mathbf{x}_0 + W$ where $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$.

Solution sets in $\mathbb{R}^2$ and $\mathbb{R}^3$

A point	A point
A line	A line
All of $\mathbb{R}^2$	A plane
	All of $\mathbb{R}^3$

Corollary

A general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of this system to a general solution $A\mathbf{x} = \mathbf{0}$.

Number of solutions

If A is an $m \times n$ matrix, then the following statements are equivalent

a. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
b. $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^m$.

It follows that if $A\mathbf{x} = \mathbf{0}$ has infinitely many solution then $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.

Corollary

A nonhomogeneous linear system with more unknowns than equations is either inconsistent or has infinitely many solutions.

Solving linear systems using vectors

Vector equation

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \longleftrightarrow x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$$

where $\mathbf{c}_j$ denotes the jth column vector of A

$$\mathbf{c}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m.$$

Theorem

A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ only has the trivial solution if and only if the column vectors of A are linearly independent.

The subspace of $\mathbb{R}^m$ spanned by the column vectors of an $m \times n$ matrix A is called the **column space** of A and is denoted by col(A)

$$\operatorname{col}(A) = \operatorname{span}\left\{\mathbf{c}_1(A), \mathbf{c}_2(A), \ldots, \mathbf{c}_n(A)\right\} = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}\,.$$

Theorem

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Fundamental theorem of invertible matrices (cont.)

If A is an $n \times n$ matrix, then the following statements are equivalent

a. The reduced echelon form of A is I_n.
b. A is a product of elementary matrices.
c. A is invertible.
d. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.
f. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
g. The column vectors of A are linearly independent.
h. The row vectors of A are linearly independent.

Hyperplanes

If $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ is a nonzero vector and b is a scalar, then the set of points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ that satisfy the linear equation of the form

$$\mathbf{a} \cdot \mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

is called a **hyperplane** in $\mathbb{R}^n$.

If $b = 0$, then the hyperplane passes through the origin $\mathbf{0}$ and consists of all vectors $\mathbf{x} \in \mathbb{R}^n$ orthogonal to the vector $\mathbf{a}$. Accordingly, we say that the **hyperplane through the origin with normal** $\mathbf{a}$ or the **orthogonal complement of** $\mathbf{a}$. We denote this hyperplane by the symbol $\mathbf{a}^\perp$ (read "$\mathbf{a}$ perp")

$$\mathbf{a}^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{a} = 0\}.$$

Geometric interpretations of solution space

If $\mathbf{r}_1(A), \mathbf{r}_2(A), \ldots, \mathbf{r}_m(A)$ are row vectors of A, then the solution space of the linear system $A\mathbf{x} = \mathbf{0}$ can be viewed as the intersection of these hyperplanes

$$\begin{gathered}\mathbf{r}_1(A) \cdot \mathbf{x} = 0 \\ \mathbf{r}_1(A) \cdot \mathbf{x} = 0 \\ \vdots \\ \mathbf{r}_m(A) \cdot \mathbf{x} = 0\end{gathered}$$

Thus, we have the result

Theorem

The solution space of $A\mathbf{x} = \mathbf{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.