SF1684 Algebra and Geometry

Lecture 1

Vectors, lines and planes, linear equations

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Outline

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Euclidean space

Euclidean n-space is the set of all **ordered n-tuples** of real numbers

Euclidean n-space

$$\mathbb{R}^n = \left\{(x_1, x_2, \ldots, x_n) \;\middle|\; x_i \in \mathbb{R}, 1 \leq i \leq n\right\}.$$

An element $\mathbf{x} \in \mathbb{R}^n$ is called a **point** or a **vector** (depending on the context). We write

$$\mathbf{x} = (x_1, \ldots, x_n) \qquad \text{or} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

where $x_1, \ldots, x_n$ are called the coordinates (or components) of $\mathbf{x}$.

Definitions

Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ are called equal if $u_i = v_i$ for all $1 \leq i \leq n$.

The zero vector $\mathbf{0} \in \mathbb{R}^n$ is defined by $\mathbf{0} = (0, 0, \ldots, 0)$.

The negative of $\mathbf{u}$ is defined by $-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$.

Algebraic operations

For $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ and $k \in \mathbb{R}$:

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \\
k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n), \\
-\mathbf{u} &= (-1)\mathbf{u} = (-u_1, -u_2, \ldots, -u_n), \\
\mathbf{u} - \mathbf{v} &= \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n).
\end{aligned}$$

Algebraic properties

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $k, l \in \mathbb{R}$ then

i. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ [Commutative]
ii. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ [Associative]
iii. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ [Additive identity]
iv. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ [Additive inverse]
v. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$ [Distributive]
vi. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ [Distributive]
vii. $k(l\mathbf{u}) = (kl)\mathbf{u}$ [Associative]
viii. $1\mathbf{u} = \mathbf{u}$ [Multiplicative identity]

Parallel (collinear) vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} = k\mathbf{v}$ for some scalar k.

Linear combinations

Linear combination

A vector $\mathbf{u} \in \mathbb{R}^n$ is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there are scalars $c_1, c_2, \dots, c_k \in \mathbb{R}$ so that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \sum_{i=1}^{k} c_i\mathbf{v}_i.$$

The scalars $c_1, c_2, \dots, c_k$ are called the coefficients of this linear combination.

We define the standard unit vectors in $\mathbb{R}^n$ to be

$$\mathbf{e}_1 = (1, 0, \dots, 0),\ \mathbf{e}_2 = (0, 1, \dots, 0),\ \dots,\ \mathbf{e}_n = (0, 0, \dots, 1).$$

Then every vector $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ is a linear combination of the standard unit vectors

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_n\mathbf{e}_n.$$

Matrices

Matrix

A matrix is a rectangular array of real (or complex) numbers called the **entries**, or **elements**, of the matrix. Let A be a matrix having m rows and n columns. Then A is said to have **size** $\boldsymbol{m} \times \boldsymbol{n}$ and can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

so that a_{ij} is the entry in row i and column j of A.

Dot products

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$.

Dot product (Euclidean inner product)

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Length (Euclidean norm or magnitude)

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Distance between two points

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

Theorem

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ then

i. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
ii. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
iii. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
iv. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Theorem

If $\mathbf{u} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ then

i. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
ii. $\|k\mathbf{u}\| = |k| \, \|\mathbf{u}\|$

Cauchy-Schwarz inequality

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Triangle inequality

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \, .$$

Angle between vectors

The angle θ between two nonzero vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the **smallest angle** that one vector can be rotated counterclockwise to align with the other.

Theorem

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \qquad \text{or} \qquad \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right).$$

Corollary

Two vectors (in $\mathbb{R}^2$ or $\mathbb{R}^3$) are perpendicular if and only if their dot product is zero.

Orthogonality

Definitions

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are called **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is called orthogonal if every pair of vectors is orthogonal, that is

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0, \quad \text{for all } i \neq j.$$

It is **orthonormal** if it is orthogonal, and in addition $\|\mathbf{u}_i\| = 1$ for all $i = 1, \ldots, k$.

Pythagorean theorem in $\mathbb{R}^n$

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in $\mathbb{R}^n$, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Lines

The vector equation of a straight line through a given point $\mathbf{x}_0$ in the direction of a known vector $\mathbf{v}$ has form

Vector equation

$$\mathbf{x} - \mathbf{x}_0 = t\mathbf{v} \qquad \text{or} \qquad \mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{$*$}$$

with parameter $t \in \mathbb{R}$.

By equating corresponding components we can obtain the parametric equations of the line. For example in $\mathbb{R}^3$, if we let $\mathbf{x}_0 = (x_0, y_0, z_0)$, $\mathbf{x} = (x, y, z)$ and $\mathbf{v} = (a, b, c)$ then $(*)$ becomes

Parametric equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \, , \\ z = z_0 + ct \end{cases} \quad \text{for } t \in \mathbb{R}.$$

Lines

Vector equation

The vector equation of the line passing through two distinct points $\mathbf{x}_0$ and $\mathbf{x}_1$ is

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \qquad \text{or} \qquad \mathbf{x} = (1-t)\mathbf{x}_0 + t\mathbf{x}_1.$$

Planes

General equation (scalar or Cartesian equation) of a plane

$$Ax + By + Cz = D.$$

The **normal vector** to a plane is a vector which is perpendicular (orthogonal) to the plane. The point-normal equation of the plane through $\mathbf{x}_0 = (x_0, y_0, z_0)$ with normal $\mathbf{n} = (a, b, c)$ is

Point-normal equation

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

Planes

The vector equation of the plane through point $\mathbf{x}_0$ that is parallel to two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (nonzero and noncollinear) is

Vector equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

with parameters $t_1, t_2 \in \mathbb{R}$.

If we let $\mathbf{x}_0 = (x_0, y_0, z_0)$, $\mathbf{x} = (x, y, z)$ and $\mathbf{v}_1 = (a_1, b_1, c_1)$, $\mathbf{v}_2 = (a_2, b_2, c_2)$ then we can obtain the parametric equations of the plane

Parametric equations

$$\begin{cases} x = x_0 + a_1t_1 + a_2t_2 \\ y = y_0 + b_1t_1 + b_2t_2 \\ z = z_0 + c_1t_1 + c_2t_2 \end{cases}, \quad \text{for } t_1, t_2 \in \mathbb{R}.$$

Outline

Linear equations

Linear equation of n variables

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \quad a_i, b \in \mathbb{R} \text{ (or } \mathbb{C}).$$

Geometric interpretation

A linear equation of n variables represents a hyperplane (flat subspace of dimension $n-1$) in the n-dimensional Euclidean space $\mathbb{R}^n$.

A system of linear equations is a finite collection of linear equations involving the same set of variables.

System of linear equations (linear system)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Solution set

Solution of a linear system

A solution of a linear system is a n-tuple $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Theorem

Given any system of linear equations, there are three possibilities for the solution:

1. no solution (inconsistent),
2. a unique solution (consistent),
3. infinitely many solutions (consistent).

Matrices of a linear system

Coefficient $m \times n$ matrix and $m \times 1$ column vector

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \longleftrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented $m \times (n+1)$ matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Solving a linear system

Equivalent linear systems

Two linear systems are equivalent $\longleftrightarrow$ both have same solution set.

When solving a linear system, one should rewrite the system into a **simpler equivalent** system. This process consists of two parts: **elimination** and **back substitution**. Both parts can achieved using elementary row operations applied to the corresponding augmented matrix.

Elementary row operations

1. $R_i \leftrightarrow R_j$ Swap row i and row j.
2. $R_i \mapsto cR_i$ Replace row i with $c \neq 0$ times row i.
3. $R_i \mapsto R_i + cR_j$ Replace row i with the sum of row i and c times row j.

Example

$$\begin{cases} x+y-\ \ z=2 \\ 2x+y+\ \ z=6 \\ -x-y+3z=0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 6 \\ -1 & -1 & 3 & 0 \end{array}\right]$$

$$\Big\downarrow \begin{array}{c} R_2 \mapsto R_2 + (-2)R_1 \\ R_3 \mapsto R_3 + R_1 \end{array}$$

$$\begin{cases} x+y-\ \ z=3 \\ -y+3z=0 \\ 2z=2 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\Big\downarrow \begin{array}{c} R_2 \mapsto (-1)R_2 \\ R_3 \mapsto (1/2)R_3 \end{array}$$

$$\begin{cases} x+y-\ \ z=3 \\ y-3z=0 \\ z=1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Example (cont.)

$$\begin{cases} x + y - z = 3 \\ \phantom{x+{}} y - 3z = 0 \\ \phantom{x+y-3{}} z = 1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{c} R_2 \mapsto R_2 + 3R_3 \\ R_1 \mapsto R_1 + R_3 \end{array}$$

$$\begin{cases} x + y = 4 \\ \phantom{x+{}} y = 3 \\ \phantom{x+{}} z = 1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_1 \mapsto R_1 + (-1)R_2$$

$$\begin{cases} x = 1 \\ y = 3 \\ z = 1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Outline

Echelon forms

Leading entry

The leading entry of a row is the **first nonzero entry** in the row. A row without a leading entry is a row of zeros.

Row echelon form

A matrix is said to be in echelon form provided that

1. The leading entry of every nonzero row is $\mathbf{1}$.
2. All zero rows are at the bottom of the matrix.
3. Leading 1's **shift to the right** as we go down the rows.

Reduced row echelon form

A matrix is in reduced echelon form if it is in echelon form and every other entry of a column which contains a leading 1 is **zero**.

Echelon forms

$$\begin{bmatrix} \boxed{1} & * & * & * & * \\ 0 & \boxed{1} & * & * & * \\ 0 & 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix} \qquad \begin{bmatrix} 0 & \boxed{1} & * & * & * & * & * & * & * \\ 0 & 0 & \boxed{1} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \boxed{1} & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced echelon forms

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix} \qquad \begin{bmatrix} 0 & \boxed{1} & 0 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & \boxed{1} & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gauss-Jordan elimination

The goal of the Gauss-Jordan elimination is to convert an augmented matrix into reduced row echelon form. This process splits into two parts:

1. The forward phase (Gaussian elimination): zero-out all entries below leading 1's, from left to right $\rightarrow$ **row echelon form**.
2. The backward phase: zero-out all entries above the leading 1's, from right to left.

Example

Solve the following system

$$\begin{cases} \phantom{4x+{}} 6y + 2z = 3 \\ 4x + 12y - 8z = 6 \\ 4x + 6y - 10z = 3 \end{cases}$$

Example (cont.)

$$\left[\begin{array}{ccc|c} 0 & 6 & 2 & 3 \\ 4 & 12 & -8 & 6 \\ 4 & 6 & -10 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 4 & 12 & -8 & 6 \\ 0 & 6 & 2 & 3 \\ 4 & 6 & -10 & 3 \end{array}\right] \xrightarrow{R_1 \mapsto (1/4)R_1}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 3/2 \\ 0 & 6 & 2 & 3 \\ 4 & 6 & -10 & 3 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 + (-4)R_1} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 3/2 \\ 0 & 6 & 2 & 3 \\ 0 & -6 & -2 & -3 \end{array}\right] \xrightarrow{R_2 \mapsto (1/6)R_2}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 3/2 \\ 0 & \boxed{1} & 1/3 & 1/2 \\ 0 & -6 & -2 & -3 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 + 6R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 3/2 \\ 0 & \boxed{1} & 1/3 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - 3R_2}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -3 & 0 \\ 0 & \boxed{1} & 1/3 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x - 3z = 0 \\ y + \frac{1}{3}z = \frac{1}{2} \end{cases}$$

Solution set:

$$\boxed{x = 3t, \quad y = \frac{1}{2} - \frac{1}{3}t, \quad z = t}$$

Forward phase (Gaussian elimination)

Let A be the augmented matrix of a linear system of variables.

1. Interchange rows in A so that the leftmost nonzero entry of A is in the first row. Multiply the first row by the reciprocal of the leading entry in order to introduce the leading 1.
2. Add multiples of the first row to lower rows so that there are only zeros below the leading 1 of the first row.
3. Cover the first row in A, now you have a smaller matrix B. Repeat Steps 1, 2 and 3 with B until you run out of matrix!

Backward phase

4. Starting at the last nonzero row and working upward, add multiples of each row to upper rows so that there are only zeros above the leading 1's.

General linear system in row echelon form

$$\left[\begin{array}{cccccccc|c} \boxed{1} & 0 & \boxed{*} & 0 & 0 & \boxed{*} & \boxed{*} & 0 & * \\ 0 & \boxed{1} & \boxed{*} & 0 & 0 & \boxed{*} & \boxed{*} & 0 & * \\ 0 & 0 & 0 & \boxed{1} & 0 & \boxed{*} & \boxed{*} & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{1} & \boxed{*} & \boxed{*} & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- $\boxed{1}$ are **leading variables**
- $\boxed{*}$ are **free variables**

Reading the solutions from the row echelon form

There are three possibilities for the reduced row echelon form of the augmented matrix of linear system

1. The last column has a leading $1 \rightarrow$ no solution.
2. Every column, except the last one, has a leading $1 \rightarrow$ unique solution.
3. The last column and some other columns do not have leading 1's $\rightarrow$ infinitely many solution; free variables correspond to columns without leading 1's.

Homogeneous linear system

Trivial solution

A homogeneous linear system of m equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

The system always has the trivial solution

$$x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_n = 0.$$

Remark

The reduced row echelon form of homogeneous system does not have leading 1 in the last column.

Theorem

Every homogeneous linear system has one trivial solution or infinitely many solutions.

Since each nonzero row has a leading 1 and each leading 1 corresponds to a leading variable, we have the result

Dimension theorem for homogeneous systems

If a homogeneous linear system has n unknowns and its reduced row echelon form has r nonzero rows then the system has $n - r$ free variables.

Corollary

If a homogeneous linear system has more unknowns than equations then the system has infinitely many solutions.