Department of Mathematics

SF1624/SF1684
Algebra and Geometry
Year 2018/2019

Problems for Seminar 4

Check the canvas page of the course for information on how seminars are run and what you are expected to do before and during the seminars.

The seminar starts with a test. The problem will be about determining matrix representation of geometrically defined transformation.

In the seminar, the following problems will be discussed.

Problem 1. (a) Determine eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}.$$

(b) Explain why the matrix $B = \frac{1}{19}A$ has the same eigenvectors as the matrix A.

Problem 2. This exercise is about linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ and their standard matrices.

(a) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the anticlockwise rotation around the origin by an angle of 90° ($\pi/2$ radians). Determine the standard matrix A for T_1.
(b) Let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = -x$. Determine the standard matrix B for T_2.
(c) Determine the standard matrix C for the composition $T_2 \circ T_1$.
(d) The map $T_2 \circ T_1$ is a reflection. About which line? Explain your answer.

Problem 3. The linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$T\left(\begin{bmatrix} 5 \\ 10 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 11 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(a) Determine the matrix A associated to T.
(b) Verify that A is its own inverse.

Problem 4. The linear map $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ x - 2y \\ y - x \end{bmatrix}.$$

(a) Find the matrix for T.

(b) Determine if one of the vectors

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \quad \text{or} \quad \vec{w} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

are in the range $\mathrm{im}(T)$.

Problem 5. A map f is sought that satisfies

$$f\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad f\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad f\left(\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Why is there no such map that is linear?

(b) How must we change the last value, $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, such that such a linear map f exists?

(c) Find the standard matrix for f (with the last value corrected as in b).

Miscellaneous

Here are some other topics that are important and interesting to discuss.

- What is the geometric significance of eigenvectors of a linear map? For instance for the eigenvalues 0 or 1?
- Why are eigenvectors for different eigenvalues linearly independent?
- What is a linear map? Is every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ a matrix map? How many rows and columns does the matrix of the map have?
- What is the nullspace/kernel and the image of a map? Which vectors constitute the nullspace and column space of a rotation and of a reflection?