Tutorial 6: FEM for Engineering Applications (SE1025)

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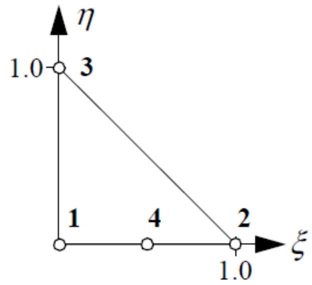
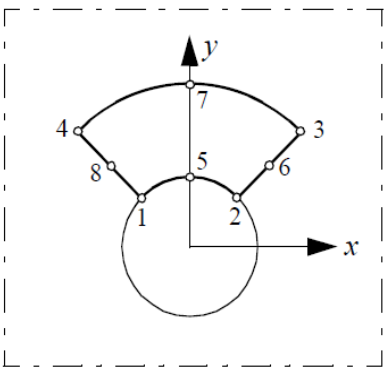
6.2 One way to satisfy compatibility across element boundaries between regions of high to low order elements is to use transition elements. A plane triangular transition element is shown in Figure 4. The shape functions for the vertex nodes of the element are displayed below the element. Determine the shape function associated with node 4 and show that it fulfil standard requirements put on shape functions.



$$N_1 = 1 - \xi(3 - 2(\xi + \eta)) - \eta$$
$$N_2 = \xi(2(\xi + \eta) - 1)$$
$$N_3 = \eta$$

6.3 A plate containing a circular hole with radius R is modelled by use of plane 8-noded bi-quadratic isoparametric elements. The figure to the right shows one such element located at the hole of the plate. The element is symmetrically located with respect to the y-axis and extends one quarter of the circumference of the hole, i.e. the straight element sides: 2-6-3 and 1-8-4, respectively, form 45° angles with respect to the y-axis. The nodes 1, 5 and 2 are placed at the border of the hole. Determine the distance from the centre of the hole to the point $\{x_0, y_0\}$, defined by the natural coordinates $\xi = \xi_0 = 1/\sqrt{2}$, $\eta = \eta_0 = -1$, i.e. calculate $\sqrt{(x_0^2 + y_0^2)}$. How much does this point deviate from the geometric boundary (the radius) of the circular hole?



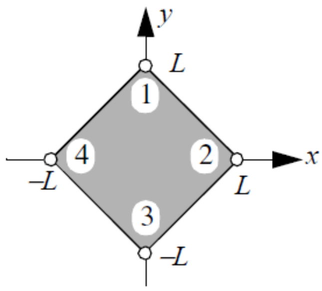
The shape functions of the element are:

$$N_1 = -\tfrac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \quad , \quad N_2 = -\tfrac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \quad ,$$

$$N_3 = -\tfrac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \quad , \quad N_4 = -\tfrac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \quad ,$$

$$N_5 = \tfrac{1}{2}(1-\xi^2)(1-\eta) \quad , \quad N_6 = \tfrac{1}{2}(1+\xi)(1-\eta^2) \quad , \quad N_7 = \tfrac{1}{2}(1-\xi^2)(1+\eta) \quad , \quad N_8 = \tfrac{1}{2}(1-\xi)(1-\eta^2) \quad .$$

6.4 The Figure to the right shows a plane bi-linear isoparametric 4-node element. The order of node numbering is opposite the one used in the natural coordinate system (ξ, η). As a consequence, the determinant of the Jacobi matrix becomes negative. The element stiffness matrix, which is calculated by area integration in the natural coordinate system, will then also become negative. A FEM-analysis with such an element will "crash". Calculate the determinant of the Jacobi matrix for the element with the erroneous node numbering to the right.

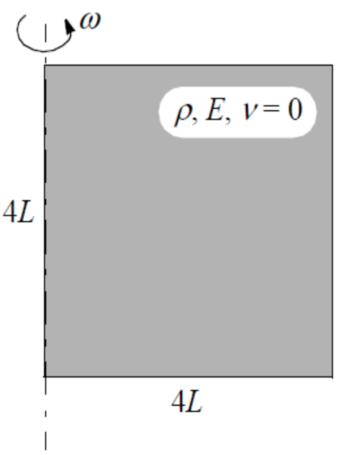


6.19 A rectangular plate of thickness h rotates with a constant angular velocity ω, see Figure (a) below. The material of the plate is isotropic linear elastic with elasticity modulus E, Poisson's ratio ν and has density ρ. Plane stress is assumed to prevail in the plate and that $\nu = 0$. If the symmetry of the problem is considered and utilized, the plate can be modelled by only two bi-linear isoparametric elements according to Figure (b) below. The node coordinates is evident from the figure. The inertia forces due to the angular velocity ω can be addressed by introducing the body forces $K_x = \rho\omega^2 x$ and $K_y = 0$ into the FEM analysis.

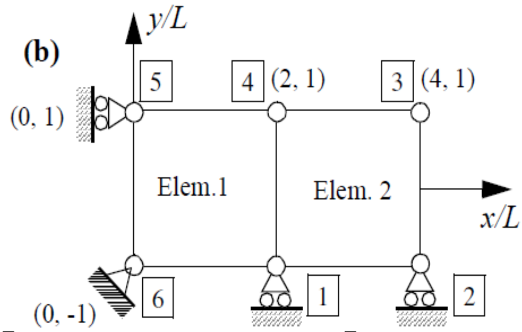
(a) Determine the contribution from the body force to the node force vector in element 2.
Hint: the coordinate transformation $x = (3 + \xi)L$, $y = \eta L$ *is useful in element 2.*

(b) The resultant node displacements, $\mathbf{D}^T$, corresponding to the current load is given in the figure below. Calculate the stresses at the three points: $\{x = 0, y = 0\}$, $\{x = L, y = 0\}$, $\{x = 2L, y = 0\}$. The exact solution for the normal stress in the x-direction can be expressed as $\sigma_{xx}(x) = (\rho\omega^2 L^2/2)(16 - (x/L)^2)$. How much does the FEM solution deviates from the exact solution at the three points? At which point do the solutions deviate the least?
Hint: the Jacobi matrix of the coordinate transformation is $\mathbf{J} = L\,\mathbf{I}$, *where* $\mathbf{I}$ *is a unit matrix of dimension 2.*

(a)



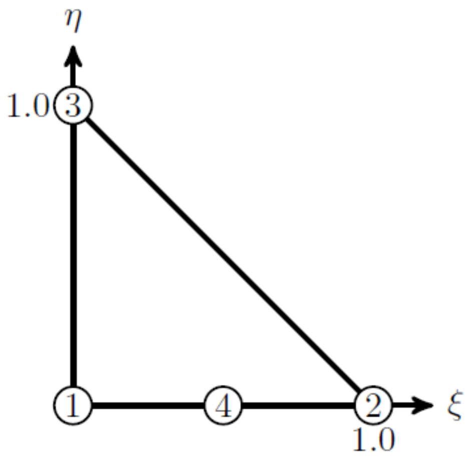
(b)

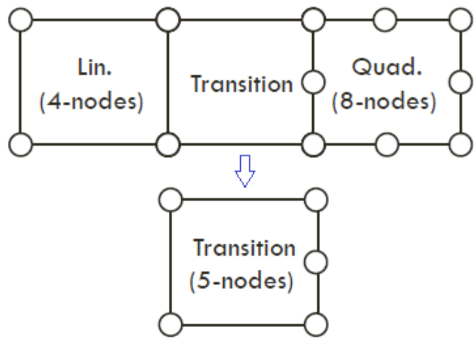


$$\mathbf{D}^T = \begin{bmatrix} D_{1x} & D_{1y} & D_{2x} & \dots & D_{6x} & D_{6y} \end{bmatrix}$$

$$= \frac{4}{3}\frac{\rho\omega^2 L^3}{E}\begin{bmatrix} 11 & 0 & 16 & 0 & 16 & 0 & 11 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 6.2





$$N_1(\xi,\eta) = 1 - \xi(3 - 2(\xi+\eta)) - \eta$$
$$N_2(\xi,\eta) = \xi(2(\xi+\eta) - 1)$$
$$N_3(\xi,\eta) = \eta$$

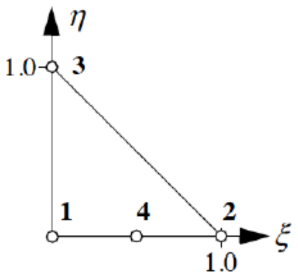
Recall Important Characteristic of Shape Functions:

1) Interpolation Condition
   The shape function has a unit value at node -*i* and zero in all other nodes
2) Local Support Condition
   Vanishes over any element boundary which does not include node -*i*
3) Inter-element Compatibility Condition
   Satisfies C_0 continuity between adjacent elements over any element boundary that includes node -*i*
4) Completeness Condition
   The interpolation is able to represent any displacement field which is a linear polynomial in X and Y.
5) Partition of units properties
   The sum of all the shape functions in each point must be 1

We know all the shape functions except for one, so we can use the last property

$$N_1 = 1 - \xi\big(3 - 2(\xi+\eta)\big) - \eta$$
$$N_2 = \xi(2(\xi+\eta) - 1)$$
$$N_3 = \eta$$

$$N_1 + N_2 + N_3 + N_4 = 1$$
$$\Leftrightarrow N_4 = 1 - N_1 - N_2 - N_3$$
$$\Leftrightarrow N_4 = 1 - \big(1 - \xi\big(3 - 2(\xi + \eta)\big) - \eta\big) - \big(\xi\big(2(\xi + \eta) - 1\big)\big) - (\eta)$$
$$\Leftrightarrow N_4 = (\ldots) =$$
$$\Leftrightarrow \boxed{N_4 = 4\xi(1 - \xi - \eta)}$$



Node (i)	(ξ_i, η_i)
1	(0,0)
2	(1,0)
3	(0,1)
4	(0.5,0)

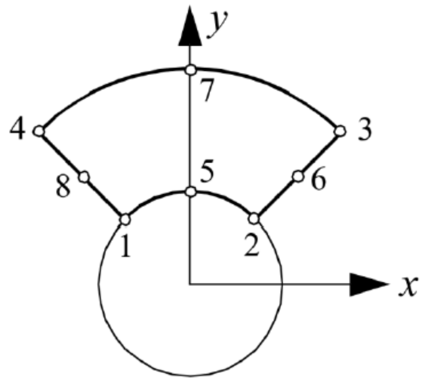
Properties 1 and 2 Satisfied

$$N_4(\xi = 0, \eta = 0) = 4 \cdot 0 \cdot (1 - 0 - 0) = 0$$
$$N_4(\xi = 1, \eta = 0) = 4 \cdot 1 \cdot (1 - 1 - 0) = 0$$
$$N_4(\xi = 0, \eta = 1) = 4 \cdot 0 \cdot (1 - 0 - 1) = 0$$
$$N_4(\xi = 0.5, \eta = 0) = 4 \cdot 0.5 \cdot (1 - 0.5 - 0) = 1$$

OK!

Exercise 6.3

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \qquad N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$$
$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \qquad N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$
$$N_3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \qquad N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)$$
$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \qquad N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$



Find the distance from the origin of the node :

$$(x_0, y_0) \quad \Rightarrow \quad \left(\xi = \xi_0 = \frac{1}{\sqrt{2}}, \eta = \eta_0 = -1\right)$$

We have to transform the local coordinates to global coordinates:

We know the definition of global coordinates (x,y) from the nodal global coordinates (x_i, y_i) and the shape functions:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_i N_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

We compute the shape function in the point of interest $\left(\xi = \xi_0 = \frac{1}{\sqrt{2}}, \eta = \eta_0 = -1\right)$

And we need the global coordinates of the nodes (x_i, y_i) for the interpolation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_i N_i(1/\sqrt{2}, -1) \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

First we compute the shape functions at the node of interest:

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) = -\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)(1--1)\left(1+\frac{1}{\sqrt{2}}+-1\right) = \frac{1}{4}-\frac{\sqrt{2}}{4}$$
$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) = -\frac{1}{4}\left(1+\frac{1}{\sqrt{2}}\right)(1--1)\left(1-\frac{1}{\sqrt{2}}+-1\right) = \frac{1}{4}+\frac{\sqrt{2}}{4}$$
$$N_3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) = -\frac{1}{4}\left(1+\frac{1}{\sqrt{2}}\right)(1+-1)\left(1-\frac{1}{\sqrt{2}}--1\right) = 0$$
$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) = 0$$
$$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta) = \frac{1}{2}\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)(1--1) = \frac{1}{2}$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2) = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)(1-(-1)^2) = 0$$
$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta) = \frac{1}{2}\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)(1+-1) = 0$$
$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2) = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)(1-(-1)^2) = 0$$

Only N1, N2, N5 are non-zero, so for the interpolation we only need the global coordinates of nodes 1,2,5.

We know from the geometry the global coordinates of nodes 1, 2, 5:

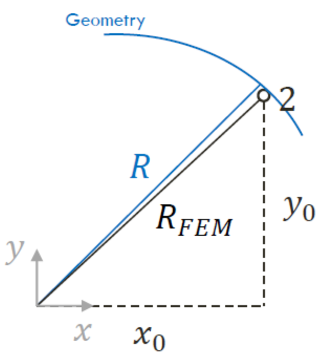
n	x_i	y_i
1	$-R/\sqrt{2}$	$R/\sqrt{2}$
2	$R/\sqrt{2}$	$R/\sqrt{2}$
5	0	R

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \sum_i N_i(1/\sqrt{2},-1)\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \frac{1-\sqrt{2}}{4}\begin{bmatrix} -R/\sqrt{2} \\ R/\sqrt{2} \end{bmatrix} + \frac{1+\sqrt{2}}{4}\begin{bmatrix} R/\sqrt{2} \\ R/\sqrt{2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ R \end{bmatrix} \quad = R\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}+2}{4} \end{bmatrix}$$

We can compute now the distance from the center

$$d = R\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}+2}{4}\right)^2}$$

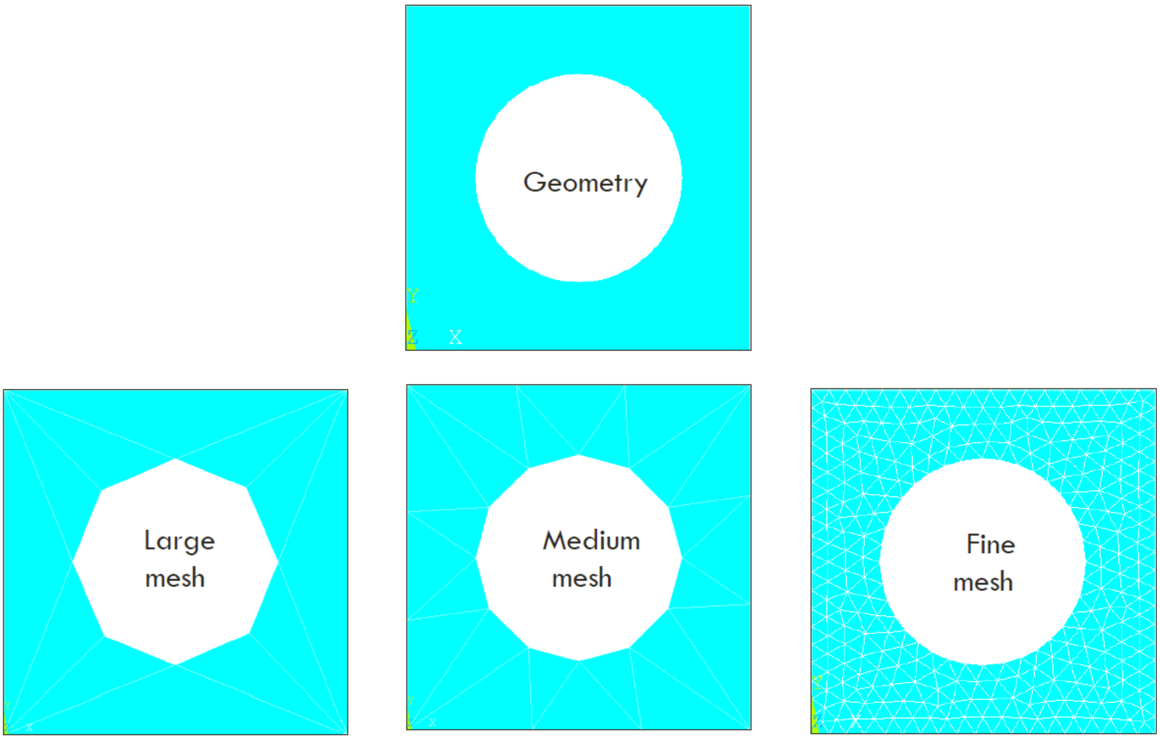
Compare with the real geometry:



$$\Rightarrow R_{FEM} = \sqrt{x_0^2 + y_0^2} =$$
$$\sqrt{\left(\frac{R}{2}\right)^2 + \left(\frac{R\sqrt{2}}{4} + \frac{R}{2}\right)^2} =$$
$$\Leftrightarrow \boxed{R_{FEM} \approx 0.989 \cdot R}$$

NOTE:

The geometry is approximated through the mesh and their shape functions:



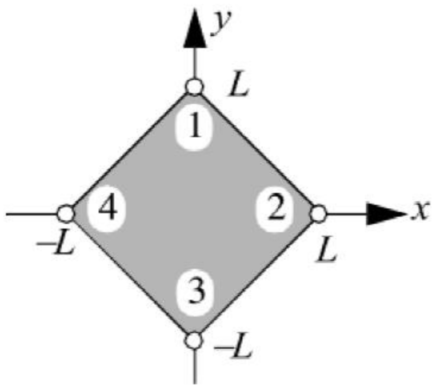
12 elements

24 elements

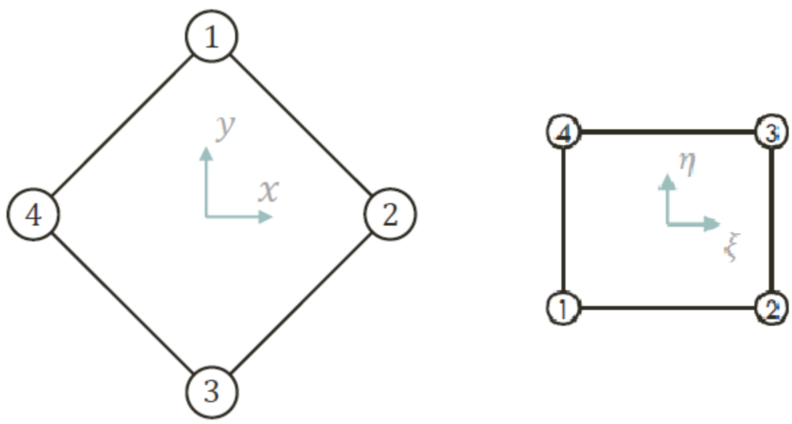
544 elements

Exercise 6.4

Importance of node ordering:



GLOBAL LOCAL



Clockwise Anti-clockwise

Compute the determinant of the Jacobian:

$$\boxed{\det(\mathbf{J})}$$

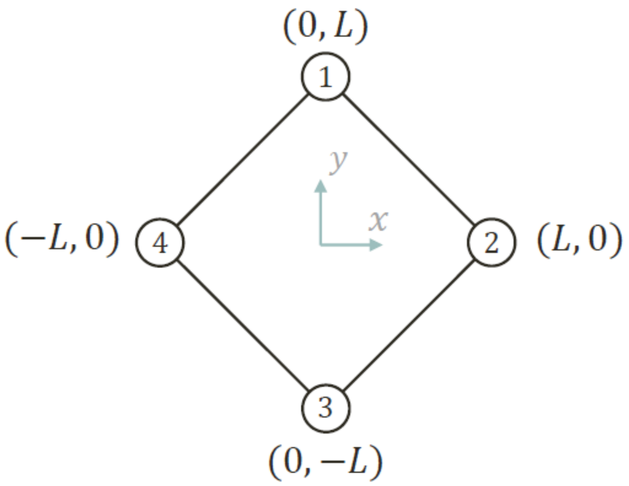
$$\boxed{\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}$$

$$\boxed{\begin{aligned} x &= \sum_i^4 N_i x_i \\ y &= \sum_i^4 N_i y_i \end{aligned}}$$

$$\boxed{x_i(x, y), y_i(x, y)}$$

$$\boxed{\begin{aligned} N_1 &= (1-\xi)(1-\eta)/4 \\ N_2 &= (1+\xi)(1-\eta)/4 \\ N_3 &= (1+\xi)(1+\eta)/4 \\ N_4 &= (1-\xi)(1+\eta)/4 \end{aligned}}$$

Step 1:



$\Rightarrow$

From the geometry:

$$\boldsymbol{x}_1 = (x_1, y_1) = (0, L)$$
$$\boldsymbol{x}_2 = (x_2, y_2) = (L, 0)$$
$$\boldsymbol{x}_3 = (x_3, y_3) = (0, -L)$$
$$\boldsymbol{x}_4 = (x_4, y_4) = (-L, 0)$$

From the shape functions:

$$N_1 = (1-\xi)(1-\eta)/4$$
$$N_2 = (1+\xi)(1-\eta)/4$$
$$N_3 = (1+\xi)(1+\eta)/4$$
$$N_4 = (1-\xi)(1+\eta)/4$$

Step 2:

$$x = \sum_i^4 N_i x_i = \frac{(1-\xi)(1-\eta)}{4}\cdot(0) + \frac{(1+\xi)(1-\eta)}{4}\cdot(L) + \frac{(1+\xi)(1+\eta)}{4}\cdot(0) + \frac{(1-\xi)(1-\eta)}{4}\cdot(-L) = \frac{L}{2}(\xi-\eta)$$

$$y = \sum_i^4 N_i y_i = \frac{(1-\xi)(1-\eta)}{4}\cdot(L) + \frac{(1+\xi)(1-\eta)}{4}\cdot(0) + \frac{(1+\xi)(1+\eta)}{4}\cdot(-L) + \frac{(1-\xi)(1-\eta)}{4}\cdot(0) = \frac{L}{2}(-\xi-\eta)$$

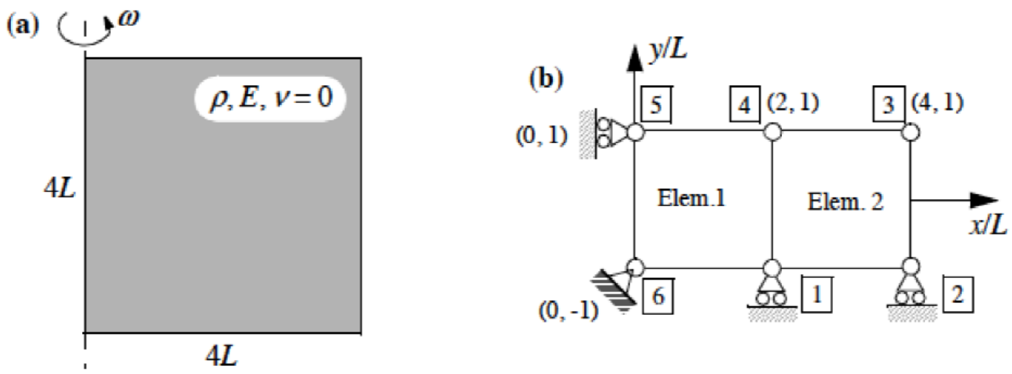
Step 3:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\left(\frac{L}{2}(\xi-\eta)\right)}{\partial \xi} & \dfrac{\partial\left(\frac{L}{2}(-\xi-\eta)\right)}{\partial \xi} \\ \dfrac{\partial\left(\frac{L}{2}(\xi-\eta)\right)}{\partial \eta} & \dfrac{\partial\left(\frac{L}{2}(-\xi-\eta)\right)}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{L}{2} & -\dfrac{L}{2} \\ -\dfrac{L}{2} & -\dfrac{L}{2} \end{bmatrix}$$

Step 4:

$$\det(\mathbf{J}) = \begin{vmatrix} \dfrac{L}{2} & -\dfrac{L}{2} \\ -\dfrac{L}{2} & -\dfrac{L}{2} \end{vmatrix} = \left(\left(\frac{L}{2}\right)\cdot\left(-\frac{L}{2}\right) - \left(-\frac{L}{2}\right)\cdot\left(-\frac{L}{2}\right)\right) = -\frac{L^2}{4} - \frac{L^2}{4} = -\frac{L^2}{2}$$

Exercise 6.19



a) Determine the contribution from the body force to the node force vector in ELEMENT 2

Body Force Data:

$$\boldsymbol{F}_{be} = \int_{V_e} \boldsymbol{N}^T \boldsymbol{K}_b dV_e$$

$$dV = h \cdot dA_{xy} = h \cdot dx \cdot dy$$

$$dA_{xy} = |\mathbf{J}| d\xi d\eta$$

$$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

Shape Functions for the Iso-Parametric Elements:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

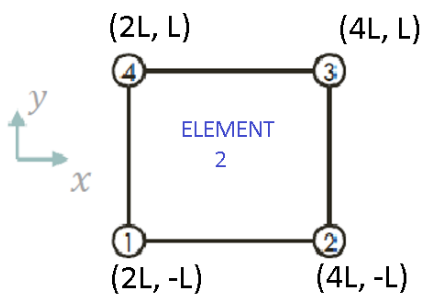
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\boldsymbol{K}_b = \begin{bmatrix} K_x \\ K_y \end{bmatrix} = \begin{bmatrix} \rho\omega^2 x \\ 0 \end{bmatrix}$$

The global coordinates are :



Coordinate Transformation

$$x = \sum_{i}^{4} N_i x_i \qquad y = \sum_{i}^{4} N_i y_i \quad \Rightarrow \quad \boxed{\begin{matrix} x = (3+\xi)L \\ y = \eta L \end{matrix}}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \Rightarrow \mathbf{J} = \begin{bmatrix} \frac{\partial\left((3+\xi)L\right)}{\partial \xi} & \frac{\partial(\eta L)}{\partial \xi} \\ \frac{\partial\left((3+\xi)L\right)}{\partial \eta} & \frac{\partial(\eta L)}{\partial \eta} \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$$

$$|\mathbf{J}| = (L \cdot L) - (0 \cdot 0) = L^2$$

So we compute the integral in LOCAL:

$$\Rightarrow \boldsymbol{F}_{b,2} = hL^2 \int_{-1}^{1}\int_{-1}^{1} \frac{1}{4} \begin{bmatrix} (1-\xi)(1-\eta) & 0 \\ 0 & (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) & 0 \\ 0 & (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) & 0 \\ 0 & (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) & 0 \\ 0 & (1-\xi)(1+\eta) \end{bmatrix} \begin{bmatrix} \rho\omega^2 \cdot \left((3+\xi)L\right) \\ 0 \end{bmatrix} d\xi d\eta$$

$$x = (3+\xi)L$$

$$\Rightarrow \boldsymbol{F}_{b,2} = \frac{\rho\omega^2 hL^3}{3} \begin{bmatrix} 8 \\ 0 \\ 10 \\ 0 \\ 10 \\ 0 \\ 8 \\ 0 \end{bmatrix}$$

Determine the stress in the following points:

(1) $x_1=0$, $y_1=0$;
(2) $x_2=L$, $y_2=0$;
(3) $x_3=2L$; $y_3=0$

The stress needs to be computed locally. All the points are in Element 1.

$$\boldsymbol{\sigma} = \boldsymbol{c}\boldsymbol{B}\boldsymbol{d}_e$$

Constitutive Matrix

$$\boldsymbol{c} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix}$$

$$\{v=0\}$$

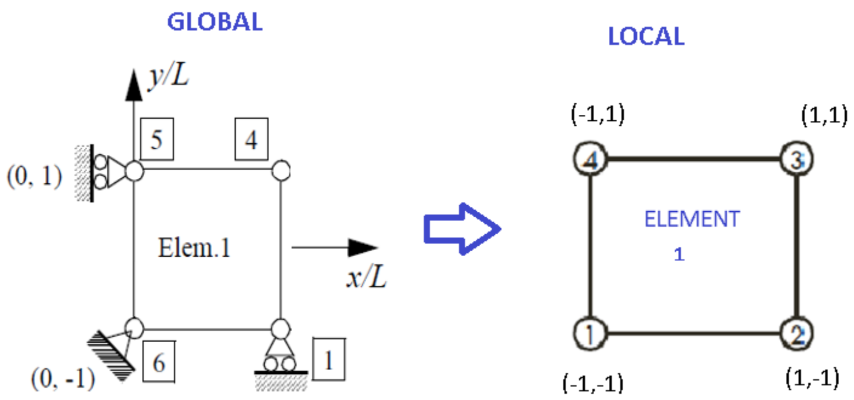
$$= E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{2} \end{bmatrix}$$

$$\begin{cases} \boldsymbol{B} = [\boldsymbol{B}_1\ \boldsymbol{B}_2\ \boldsymbol{B}_3\ \boldsymbol{B}_4] \\ \boldsymbol{B}_i = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} \end{bmatrix} \end{cases}$$

Displacement Vector
Element 1

$$\boldsymbol{d}_{e1} = \frac{4\rho\omega^2 L^3}{3E}\begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 11 \\ 0 \\ 11 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{l} 6x \\ 6y \\ 1x \\ 1y \\ 4x \\ 4y \\ 5x \\ 5y \end{array}$$

Note about the Displacement Vector de1



NODE GLOBAL		NODE LOCAL
6	⇨	1
1	⇨	2
4	⇨	3
5	⇨	4

$$
\boldsymbol{\sigma}_{e1} = E\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}_{\boldsymbol{c}}
\underbrace{\begin{array}{c}
\begin{array}{cccccccc} & & 3 & & 5 & & & \end{array} \\
\begin{bmatrix}
\frac{\partial N_1}{\partial x} & 0 & \boxed{\frac{\partial N_2}{\partial x}} & 0 & \boxed{\frac{\partial N_3}{\partial x}} & 0 & \frac{\partial N_4}{\partial x} & 0 \\
0 & \frac{\partial N_1}{\partial y} & \boxed{0} & \frac{\partial N_2}{\partial y} & \boxed{0} & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \boxed{\frac{\partial N_2}{\partial y}} & \frac{\partial N_2}{\partial x} & \boxed{\frac{\partial N_3}{\partial y}} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x}
\end{bmatrix}
\end{array}}_{\boldsymbol{B}_1}
\underbrace{\frac{4\rho\omega^2 L^3}{3E}
\begin{array}{cc}
\begin{bmatrix} 0 \\ 0 \\ \boxed{11} \\ 0 \\ \boxed{11} \\ 0 \\ 0 \\ 0 \end{bmatrix} &
\begin{array}{c} \\ \\ 3 \\ \\ 5 \\ \\ \\ \\ \end{array}
\end{array}}_{\boldsymbol{d}_1}
$$

$$
\begin{bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{bmatrix} = \left|\mathbf{J}^{-1}\right| \begin{bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{bmatrix} = \frac{1}{L}\begin{bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{bmatrix}
$$

$$
= \frac{4\rho\omega^2 L^3}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ 0 & 0 \\ \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix}\begin{bmatrix} 11 \\ 11 \end{bmatrix} = \frac{4\rho\omega^2 L^3}{3}\begin{bmatrix} \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ 0 & 0 \\ \frac{1}{2}\cdot\frac{\partial N_2}{\partial y} & \frac{1}{2}\cdot\frac{\partial N_3}{\partial y} \end{bmatrix}\begin{bmatrix} 11 \\ 11 \end{bmatrix} =
$$

$$
= \frac{4\rho\omega^2 L^3}{3}\begin{bmatrix} \frac{1}{L}\cdot\frac{\partial N_2}{\partial \xi} & \frac{1}{L}\cdot\frac{\partial N_3}{\partial \xi} \\ 0 & 0 \\ \frac{1}{2L}\cdot\frac{\partial N_2}{\partial \eta} & \frac{1}{2L}\cdot\frac{\partial N_3}{\partial \eta} \end{bmatrix}\begin{bmatrix} 11 \\ 11 \end{bmatrix} = \frac{\rho\omega^2 L^2}{3}\begin{bmatrix} 1-\eta & 1+\eta \\ 0 & 0 \\ -1-\xi & 1+\xi \end{bmatrix}\begin{bmatrix} 11 \\ 11 \end{bmatrix} = \frac{\rho\omega^2 L^2}{3}\begin{bmatrix} 22 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\boxed{\boldsymbol{\sigma}_{e1} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{\rho\omega^2 L^2}{3}\begin{bmatrix} 22 \\ 0 \\ 0 \end{bmatrix}}
$$

$$\sigma_{xx}^{Analytic}(x) = \frac{\rho\omega^2 L^2}{2}\left(16 - \left(\frac{x}{L}\right)^2\right)$$

$$\sigma_{xx}^{Analytic}(x = 0) = \frac{16}{2} \cdot \rho\omega^2 L^2$$

$$\sigma_{xx}^{Analytic}(x = L) = \frac{15}{2} \cdot \rho\omega^2 L^2$$

$$\sigma_{xx}^{Analytic}(x = 2L) = \frac{12}{2} \cdot \rho\omega^2 L^2$$

$\Rightarrow$

ERROR

$$\frac{\sigma_{xx}^{FEM} - \sigma_{xx}^{Analytic}}{\sigma_{xx}^{Analytic}}$$