Tutorial 5: FEM for Engineering Applications (SE1025)

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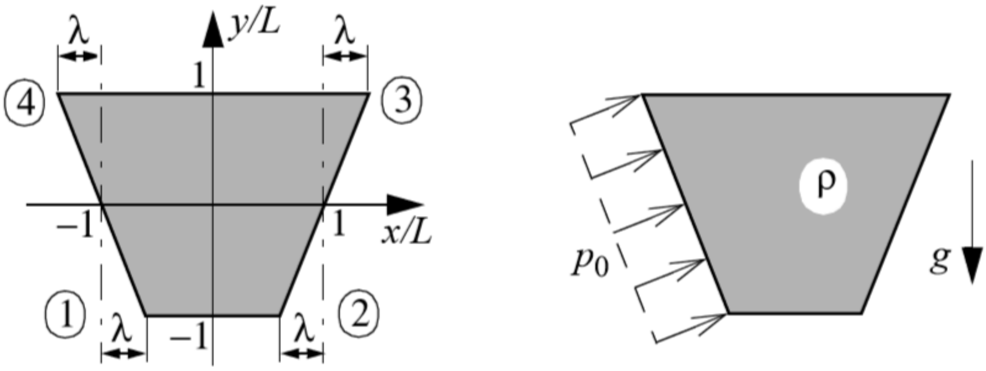
6.8 The plate in the above problem is loaded by a pressure p_0 (uniform traction) acting on the side 1-4 and by its dead weight (density ρ). The dead weight can be modelled as a force per unit volume (body force) $K_y = -\rho g$. Let $\lambda = 1/2$ and determine the contributions to the nodal force vector from

(a) the pressure p_0 and (b) the force per unit volume K_y.



6.9 Calculate the contribution from the force per unit volume K_y to the nodal force vector in the above problem by use of numerical integration based on Gauss-Legendre quadrature. Use: (a) 1×1 and (b) 2×2 point integration scheme in the element.

6.10 A traction vector $\mathbf{t}$ (force per unit surface) is acting between points A and B located on the edge of a plate of thickness h. The segment between A and B is straight and of length $2L$. Consider a linear variation of the traction vector according to

$$\mathbf{t} = \frac{1}{2}\left(1 - \frac{s}{L}\right)\mathbf{t}_A + \frac{1}{2}\left(1 + \frac{s}{L}\right)\mathbf{t}_B,$$

where s is a natural coordinate, $\mathbf{t}_A$ and $\mathbf{t}_B$ are the traction vectors at the points A and B, respectively, see the figure below. Determine the contribution to the total nodal force vector if the plate is modelled by

(a) one isoparametric 4-node quadrilateral element,

(b) one isoparametric 8-node quadrilateral element, where the mid nodes are placed in the middle between their corresponding corner nodes.

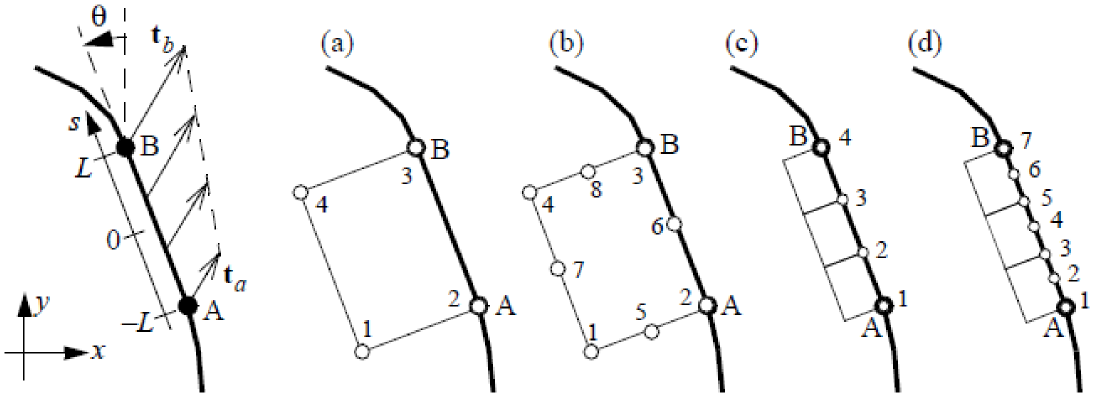
Assume that the traction vectors along AB are composed of a constant normal stress σ_0 and a constant shear stress τ_0, such that

$$\mathbf{t}_A = \mathbf{t}_B = \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sigma_0 \cos\theta - \tau_0 \sin\theta \\ \sigma_0 \sin\theta + \tau_0 \cos\theta \end{bmatrix}.$$

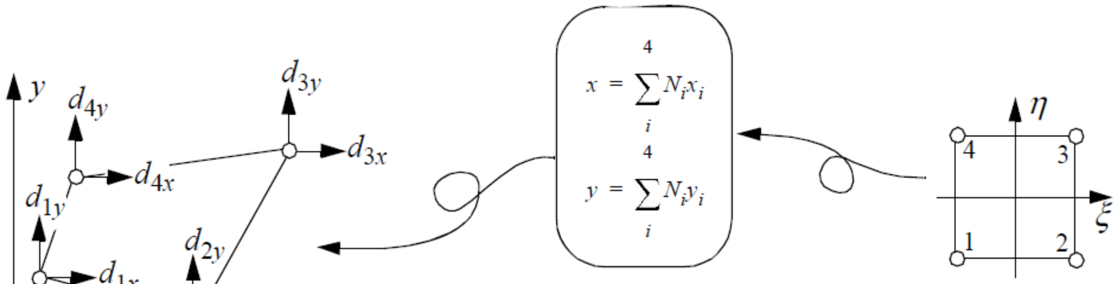
Use the results in (a) and (b) to evaluate the contribution to the total nodal force vector along AB if the boundary is modelled by

(c) three equal isoparametric 4-node quadrilateral elements and

(d) three equal isoparametric 8-node quadrilateral elements, see the figure below.



Plane (2D) quadrilateral bi-linear element:



$N_1 = (1-\xi)(1-\eta)/4, \quad N_2 = (1+\xi)(1-\eta)/4$

$N_3 = (1+\xi)(1+\eta)/4, \quad N_4 = (1-\xi)(1+\eta)/4$

Displace-ments:

$$\begin{bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \mathbf{d}_e = \mathbf{N}\mathbf{d}_e$$

Deformation:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix} \qquad \mathbf{B}_i = \begin{bmatrix} \partial N_i/\partial x & 0 \\ 0 & \partial N_i/\partial y \\ \partial N_i/\partial y & \partial N_i/\partial x \end{bmatrix}$$

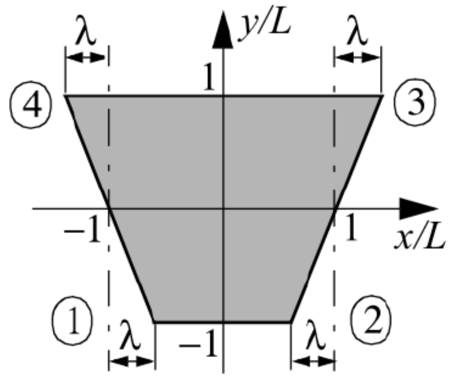
where

$$\begin{bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix}$$

Stresses:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \qquad \text{(Plane stress)}$$

Problem 6.8



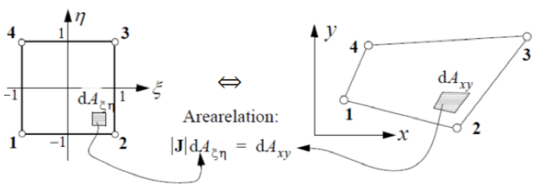
Thickness h

Density ρ

Body Force K_y= - ρg

Recall from the theory:

Element Transformation



Shape Functions

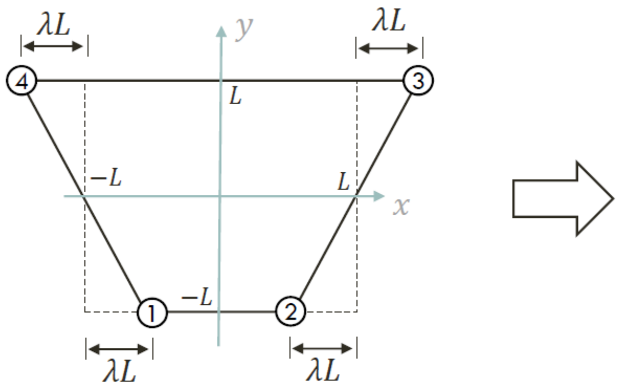
$$x = \sum_i^4 N_i x_i \qquad y = \sum_i^4 N_i y_i$$

$$N_1 = (1-\xi)(1-\eta)/4$$
$$N_2 = (1+\xi)(1-\eta)/4$$
$$N_3 = (1+\xi)(1+\eta)/4$$
$$N_4 = (1-\xi)(1+\eta)/4$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$dA_{xy} = dxdy = |\mathbf{J}|d\xi d\eta$$

Coordinates:



Node (i)	x_i	y_i
1	$-L+\lambda L$	$-L$
2	$L-\lambda L$	$-L$
3	$L+\lambda L$	L
4	$-L-\lambda L$	L

Relations between local and global reference system:

$$x = \sum_i^4 N_i x_i = \underbrace{\frac{(1-\xi)(1-\eta)}{4}}_{N_1} \cdot \underbrace{(-L+\lambda L)}_{x_1} + \underbrace{\frac{(1+\xi)(1-\eta)}{4}}_{N_2} \cdot \underbrace{(L-\lambda L\,)}_{x_2} + \underbrace{\frac{(1+\xi)(1+\eta)}{4}}_{N_3} \cdot \underbrace{(L+\lambda L\,)}_{x_3} + \underbrace{\frac{(1-\xi)(1+\eta)}{4}}_{N_4} \cdot \underbrace{(-L-\lambda L\,)}_{x_4}$$

$$\Leftrightarrow x = L\xi(1+\lambda\eta)$$

$$y = \sum_i^4 N_i y_i = \frac{(1-\xi)(1-\eta)}{4} \cdot (-L) + \frac{(1+\xi)(1-\eta)}{4} \cdot (-L\,) + \frac{(1+\xi)(1+\eta)}{4} \cdot (L) + \frac{(1-\xi)(1+\eta)}{4} \cdot (L)$$

$$\Leftrightarrow y = L\eta$$

Definition of Jacobian:

$$\boldsymbol{J}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}\frac{\partial\left(L\xi(1+\lambda\eta)\right)}{\partial \xi} & \frac{\partial (L\eta)}{\partial \xi}\\ \frac{\partial\left(L\xi(1+\lambda\eta)\right)}{\partial \eta} & \frac{\partial (L\eta)}{\partial \eta}\end{bmatrix}=\begin{bmatrix}L(1+\lambda\eta) & 0\\ L\lambda\xi & L\end{bmatrix}$$

Determinant of the Jacobian

$$|\mathbf{J}|=det\left(\begin{bmatrix}L(1+\lambda\eta) & 0\\ L\lambda\xi & L\end{bmatrix}\right)=L(1+\lambda\eta)\cdot L-L\lambda\xi\cdot 0=L^2(1+\lambda\eta)$$

$$dxdy=|\mathbf{J}|d\xi d\eta \Longrightarrow dxdy=L^2(1+\lambda\eta)d\xi d\eta$$

Matrix B:

$$\mathbf{B}=\begin{bmatrix}\mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4\end{bmatrix} \quad \text{where}$$

$$\mathbf{B}_i=\begin{bmatrix}\partial N_i/\partial x & 0\\ 0 & \partial N_i/\partial y\\ \partial N_i/\partial y & \partial N_i/\partial x\end{bmatrix} \quad \text{with} \quad \begin{bmatrix}\partial N_i/\partial x\\ \partial N_i/\partial y\end{bmatrix}=\mathbf{J}^{-1}\begin{bmatrix}\partial N_i/\partial \xi\\ \partial N_i/\partial \eta\end{bmatrix}=\frac{1}{L}\begin{bmatrix}1/(1+\lambda\eta) & 0\\ -\lambda\xi/(1+\lambda\eta) & 1\end{bmatrix}\begin{bmatrix}\partial N_i/\partial \xi\\ \partial N_i/\partial \eta\end{bmatrix}$$

$$\mathbf{J}^{-1}=\frac{1}{L}\begin{bmatrix}1/(1+\lambda\eta) & 0\\ -\lambda\xi/(1+\lambda\eta) & 1\end{bmatrix}$$

where

$$\begin{bmatrix}\partial N_1/\partial x\\ \partial N_1/\partial y\end{bmatrix}=\frac{1}{4L}\begin{bmatrix}\frac{-(1-\eta)}{1+\lambda\eta}\\ -(1-\xi)+\frac{(1-\eta)\lambda\xi}{1+\lambda\eta}\end{bmatrix} \quad \begin{bmatrix}\partial N_2/\partial x\\ \partial N_2/\partial y\end{bmatrix}=\frac{1}{4L}\begin{bmatrix}\frac{1-\eta}{1+\lambda\eta}\\ -(1+\xi)-\frac{(1-\eta)\lambda\xi}{1+\lambda\eta}\end{bmatrix}$$

$$\begin{bmatrix}\partial N_3/\partial x\\ \partial N_3/\partial y\end{bmatrix}=\frac{1}{4L}\begin{bmatrix}\frac{1+\eta}{1+\lambda\eta}\\ (1+\xi)-\frac{(1+\eta)\lambda\xi}{1+\lambda\eta}\end{bmatrix} \quad \begin{bmatrix}\partial N_4/\partial x\\ \partial N_4/\partial y\end{bmatrix}=\frac{1}{4L}\begin{bmatrix}\frac{-(1+\eta)}{1+\lambda\eta}\\ (1-\xi)+\frac{(1+\eta)\lambda\xi}{1+\lambda\eta}\end{bmatrix}$$

$$\mathbf{B}_1 = \frac{1}{4L}\begin{bmatrix} \frac{-(1-\eta)}{1+\lambda\eta} & 0 \\ 0 & -(1-\xi)+\frac{(1-\eta)\lambda\xi}{1+\lambda\eta} \\ (1+\xi)-\frac{(1+\eta)\lambda\xi}{1+\lambda\eta} & \frac{-(1-\eta)}{1+\lambda\eta} \end{bmatrix}$$

$$\mathbf{B}_2 = \frac{1}{4L}\begin{bmatrix} \frac{1-\eta}{1+\lambda\eta} & 0 \\ 0 & -(1-\xi)-\frac{(1-\eta)\lambda\xi}{1+\lambda\eta} \\ -(1+\xi)-\frac{(1+\eta)\lambda\xi}{1+\lambda\eta} & \frac{(1-\eta)}{1+\lambda\eta} \end{bmatrix}$$

$$\mathbf{B}_3 = \frac{1}{4L}\begin{bmatrix} \frac{1-\eta}{1+\lambda\eta} & 0 \\ 0 & (1-\xi)-\frac{(1-\eta)\lambda\xi}{1+\lambda\eta} \\ (1+\xi)-\frac{(1+\eta)\lambda\xi}{1+\lambda\eta} & \frac{(1-\eta)}{1+\lambda\eta} \end{bmatrix}$$

$$\mathbf{B}_4 = \frac{1}{4L}\begin{bmatrix} \frac{-(1+\eta)}{1+\lambda\eta} & 0 \\ 0 & (1-\xi)+\frac{(1+\eta)\lambda\xi}{1+\lambda\eta} \\ (1-\xi)+\frac{(1+\eta)\lambda\xi}{1+\lambda\eta} & \frac{-(1+\eta)}{1+\lambda\eta} \end{bmatrix}$$

$$\mathbf{C} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Element Stiffness Matrix

$$\mathrm{K}_e = \int_{A_e} \mathbf{B}^T\mathbf{C}\mathbf{B}h\,dA = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{C}\mathbf{B}h\,|\mathbf{J}|\,d\xi\,d\eta = \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \mathbf{B}_1^T\mathbf{C}\mathbf{B}_1 & \mathbf{B}_1^T\mathbf{C}\mathbf{B}_2 & \mathbf{B}_1^T\mathbf{C}\mathbf{B}_3 & \mathbf{B}_1^T\mathbf{C}\mathbf{B}_4 \\ \mathbf{B}_2^T\mathbf{C}\mathbf{B}_1 & \mathbf{B}_2^T\mathbf{C}\mathbf{B}_2 & \mathbf{B}_2^T\mathbf{C}\mathbf{B}_3 & \mathbf{B}_2^T\mathbf{C}\mathbf{B}_4 \\ \mathbf{B}_3^T\mathbf{C}\mathbf{B}_1 & \mathbf{B}_3^T\mathbf{C}\mathbf{B}_2 & \mathbf{B}_3^T\mathbf{C}\mathbf{B}_3 & \mathbf{B}_3^T\mathbf{C}\mathbf{B}_4 \\ \mathbf{B}_4^T\mathbf{C}\mathbf{B}_1 & \mathbf{B}_4^T\mathbf{C}\mathbf{B}_2 & \mathbf{B}_4^T\mathbf{C}\mathbf{B}_3 & \mathbf{B}_4^T\mathbf{C}\mathbf{B}_4 \end{bmatrix} hL^2(1+\lambda\eta)\,d\xi\,d\eta$$

We want to compute the dead weight:

$$\boldsymbol{F_b} = \int_{V_e} \boldsymbol{N^T K_b} dV \qquad \boldsymbol{K}_b = \begin{bmatrix} K_{bx} \\ K_{by} \end{bmatrix}$$

$$\boldsymbol{F_b} = h \int \int \boldsymbol{N^T K_b} dxdy \qquad \boldsymbol{K_b} = \begin{bmatrix} 0 \\ -\rho g \end{bmatrix}$$

Introducing the values of the shape functions:

$$\boldsymbol{F_b} = h \int \int \boldsymbol{N^T K_b} dxdy = h \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix} \frac{(1-\xi)(1-\eta)}{4} & 0 \\ 0 & \frac{(1-\xi)(1-\eta)}{4} \\ \frac{(1+\xi)(1-\eta)}{4} & 0 \\ 0 & \frac{(1+\xi)(1-\eta)}{4} \\ \frac{(1+\xi)(1+\eta)}{4} & 0 \\ 0 & \frac{(1+\xi)(1+\eta)}{4} \\ \frac{(1-\xi)(1+\eta)}{4} & 0 \\ 0 & \frac{(1-\xi)(1+\eta)}{4} \end{bmatrix} \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} L^2 (1+\lambda\eta) d\xi d\eta$$

$$\boldsymbol{F_b} = -\frac{\rho g h L^2}{4} \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix} 0 \\ (1-\xi)(1-\eta) \\ 0 \\ (1+\xi)(1-\eta) \\ 0 \\ (1+\xi)(1+\eta) \\ 0 \\ (1-\xi)(1+\eta) \end{bmatrix} (1+\lambda\eta) d\xi d\eta$$

$$\Rightarrow -\frac{\rho g h L^2}{4} \int_{-1}^{1} \int_{-1}^{1} (1 \pm \xi)(1 \pm \eta)(1+\lambda\eta) d\xi d\eta =$$

$$\Rightarrow -\frac{\rho g h L^2}{4} \underbrace{\int_{-1}^{1} (1 \pm \xi) d\xi}_{I_\xi} \underbrace{\int_{-1}^{1} (1 \pm \eta)(1+\lambda\eta) d\eta =}_{I_\eta}$$

$$\begin{cases} I_\xi = \int_{-1}^{1} (1 \pm \xi) d\xi = \left[\xi \pm \frac{\xi^2}{2}\right]_{-1}^{1} = \left(1 \pm \frac{1^2}{2}\right) - \left(-1 \pm \frac{(-1)^2}{2}\right) = 1 \pm \frac{1}{2} + 1 \mp \frac{1}{2} = 2 \\ I_\eta = \int_{-1}^{1} (1 \pm \eta)(1 + \lambda\eta) d\eta = \int_{-1}^{1} (1 + \lambda\eta \pm \eta \pm \lambda\eta^2) d\eta = \left[\eta + \frac{\lambda\eta^2}{2} \pm \frac{\eta^2}{2} \pm \frac{\lambda\eta^3}{3}\right]_{-1}^{1} = (\ldots) = 2 \pm \frac{2\lambda}{3} \end{cases}$$

$$\boxed{\left(1 + \frac{\lambda}{2} \pm \frac{1}{2} \pm \frac{\lambda}{3}\right) - \left(-1 + \frac{\lambda}{2} \pm \frac{1}{2} \pm -\frac{\lambda}{3}\right) = 1 + 1 \pm \frac{1}{2} \mp \frac{1}{2} + \frac{\lambda}{2} - \frac{\lambda}{2} \pm \frac{\lambda}{3} \pm \frac{\lambda}{3}}$$

$$\Rightarrow \boldsymbol{F}_b = -\frac{\rho g h L^2}{4} \cdot (2) \cdot \left(2 \pm \frac{2\lambda}{3}\right) = -\frac{\rho g h L^2}{3}\left(1 \pm \frac{\lambda}{3}\right)$$

$$\Rightarrow \boldsymbol{F}_b = -\frac{\rho g h L^2}{3}\left(1 \pm \frac{\frac{1}{2}}{3}\right) = -\frac{\rho g h L^2}{6}(6 \pm 1)$$

$$\boldsymbol{F}_b = -\frac{\rho g h L^2}{6}\begin{bmatrix} 0 \\ 5 \\ 0 \\ 5 \\ 0 \\ 7 \\ 0 \\ 7 \end{bmatrix}$$

Equilibrium:

$$\sum F_x = -\frac{\rho g h L^2}{6}(0 + 0 + 0 + 0) = 0$$

$$\sum F_y = -\frac{\rho g h L^2}{6}(5 + 5 + 7 + 7) = -\frac{\rho g h L^2}{6} \cdot 24 = -4\rho g h L^2$$

Dvs $-4\rho g h L^2 = -\mathrm{mg} \Leftrightarrow 4\rho h L^2 \left[\frac{kg}{m^3} \cdot m \cdot m^2\right] = m\ [kg]$ … OK!

Problem 6.9

$$\boldsymbol{F}_b = -\frac{\rho g h L^2}{4}\int_{-1}^{1}\int_{-1}^{1}\begin{bmatrix}0\\(1-\xi)(1-\eta)\\0\\(1+\xi)(1-\eta)\\0\\(1+\xi)(1+\eta)\\0\\(1-\xi)(1+\eta)\end{bmatrix}(1+\lambda\eta)d\xi d\eta$$

$$F_{1,y} = \int_{-1}^{1}\int_{-1}^{1} -\frac{\rho g h L^2}{4}(1-\xi)(1-\eta)(1+\lambda\eta)d\xi d\eta$$

$$F_{2,y} = \int_{-1}^{1}\int_{-1}^{1} -\frac{\rho g h L^2}{4}(1+\xi)(1-\eta)(1+\lambda\eta)d\xi d\eta$$

$$F_{3,y} = \int_{-1}^{1}\int_{-1}^{1} -\frac{\rho g h L^2}{4}(1+\xi)(1+\eta)(1+\lambda\eta)d\xi d\eta$$

$$F_{4,y} = \int_{-1}^{1}\int_{-1}^{1} -\frac{\rho g h L^2}{4}(1-\xi)(1+\eta)(1+\lambda\eta)d\xi d\eta$$

$$F_{1,x} = F_{2,x} = F_{3,x} = F_{4,x} = 0$$

Gauss Numerical Integration:

$$I = \int_{-1}^{1} f(x)dx = \sum_{i=1}^{m} w_i f(x_i)$$

$$\Rightarrow I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi\eta = \sum_{i=1}^{m_\eta}\sum_{j=1}^{m_\xi} w_i\, w_j f(\xi_j,\eta_i)$$

m	ξ_j	w_j	Accuracy n
1	0	2	1
2	$-1/\sqrt{3}, 1/\sqrt{3}$	1,1	3
3	$-\sqrt{0.6}, 0, \sqrt{0.6}$	5/9, 8/9, 5/9	5
4	−0.861136, −0.339981, 0.339981, 0.861136	0.347855, 0.652145, 0.652145, 0.347855	7
5	−0.906180, −0.538469, 0, 0.538469, 0.906180	0.236927, 0.478629, 0.568889, 0.478629, 0.236927	9
6	−0.932470, −0.661209, −0.238619, 0.238619, 0.661209, 0.932470	0.171324, 0.360762, 0.467914, 0.467914, 0.360762, 0.171324	11

$$
\begin{cases}
F_{1,y} = \int_{-1}^{1}\int_{-1}^{1} \overbrace{-\frac{\rho g h L^2}{4}(1-\xi)(1-\eta)(1+\lambda\eta)}^{f(\xi,\eta)}\,d\xi\,d\eta \\
F_{2,y} = \int_{-1}^{1}\int_{-1}^{1} -\frac{\rho g h L^2}{4}(1+\xi)(1-\eta)(1+\lambda\eta)\,d\xi\,d\eta \\
F_{3,y} = \int_{-1}^{1}\int_{-1}^{1} -\frac{\rho g h L^2}{4}(1+\xi)(1+\eta)(1+\lambda\eta)\,d\xi\,d\eta \\
F_{4,y} = \int_{-1}^{1}\int_{-1}^{1} -\frac{\rho g h L^2}{4}(1-\xi)(1+\eta)(1+\lambda\eta)\,d\xi\,d\eta
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
F_{1,y} = \sum_{i=1}^{m_\xi=1}\sum_{j=1}^{m_\eta=1} w_i \cdot w_j \cdot f(\xi_i,\eta_j) = (\dots) = 2\cdot 2\cdot\left(-\frac{\rho g h L^2}{4}(1-0)(1-0)\left(1+\left(\frac{1}{2}\right)\cdot 0\right)\right) = -\rho g h L^2 \\
F_{2,y} = \sum_{i=1}^{1}\sum_{j=1}^{1} w_i \cdot w_j \cdot f(\xi_i,\eta_j) = (\dots) = 2\cdot 2\cdot\left(-\frac{\rho g h L^2}{4}(1+0)(1-0)\left(1+\left(\frac{1}{2}\right)\cdot 0\right)\right) = -\rho g h L^2 \\
F_{3,y} = \sum_{i=1}^{1}\sum_{j=1}^{1} w_i \cdot w_j \cdot f(\xi_i,\eta_j) = (\dots) = 2\cdot 2\cdot\left(-\frac{\rho g h L^2}{4}(1+0)(1+0)\left(1+\left(\frac{1}{2}\right)\cdot 0\right)\right) = -\rho g h L^2 \\
F_{4,y} = \sum_{i=1}^{1}\sum_{j=1}^{1} w_i \cdot w_j \cdot f(\xi_i,\eta_j) = (\dots) = 2\cdot 2\cdot\left(-\frac{\rho g h L^2}{4}(1-0)(1+0)\left(1+\left(\frac{1}{2}\right)\cdot 0\right)\right) = = -\rho g h L^2
\end{cases}
$$

One integration Point

$$
\boxed{
\begin{array}{c}
m_\xi = m_\eta = 1 \\
\begin{cases} \xi_{i=1} = 0 & w_{i=1} = 2 \\ \eta_{j=1} = 0 & w_{j=1} = 2 \end{cases}
\end{array}
}
$$

$$
\Rightarrow \boldsymbol{F}_b = -\frac{\rho g h L^2}{6}\begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \\ 0 \\ 6 \\ 0 \\ 6 \end{bmatrix}
$$

Two integration points

m	ξ_j	w_j	Accuracy n
1	0	2	1
2	$-1/\sqrt{3}, 1/\sqrt{3}$	1,1	3
3	$-\sqrt{0.6}, 0, \sqrt{0.6}$	5/9, 8/9, 5/9	5
4	−0.861136, −0.339981, 0.339981, 0.861136	0.347855, 0.652145, 0.652145, 0.347855	7
5	−0.906180, −0.538469, 0, 0.538469, 0.906180	0.236927, 0.478629, 0.568889, 0.478629, 0.236927	9
6	−0.932470, −0.661209, −0.238619, 0.238619, 0.661209, 0.932470	0.171324, 0.360762, 0.467914, 0.467914, 0.360762, 0.171324	11

$$
\Rightarrow
\begin{cases}
\xi_{i=1} = -\frac{1}{\sqrt{3}} & w_{i=1} = 1 \\
\xi_{i=2} = \frac{1}{\sqrt{3}} & w_{i=1} = 1 \\
\eta_{j=1} = -\frac{1}{\sqrt{3}} & w_{j=1} = 1 \\
\eta_{j=2} = \frac{1}{\sqrt{3}} & w_{j=2} = 1
\end{cases}
$$

$$
\Rightarrow F_{1y} = \sum_{j=1}^{m_\eta=2}\sum_{i=1}^{m_\xi=2} w_i w_j f(\xi_i,\eta_j) = \sum_{j=1}^{m_\eta=2}\left(1\cdot w_j\cdot f\left(-\frac{1}{\sqrt{3}},\eta_j\right) + 1\cdot w_j\cdot f\left(\frac{1}{\sqrt{3}},\eta_j\right)\right)
$$

$$
= \left(\left(1\cdot 1\cdot f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + 1\cdot 1\cdot f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\right)\right) + \left(\left(1\cdot 1\cdot f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 1\cdot 1\cdot f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right)\right) =
$$

$$
\left(\left(1\cdot 1\cdot\left(\frac{\rho g h L^2(-4\sqrt{3}+9)}{36}\right) + 1\cdot 1\cdot\left(\frac{\rho g h L^2(\sqrt{3}-6)}{36}\right)\right)\right) + \left(\left(1\cdot 1\cdot\left(\frac{\rho g h L^2(-\sqrt{3}-6)}{36}\right) + 1\cdot 1\cdot\left(\frac{\rho g h L^2(4\sqrt{3}-9)}{36}\right)\right)\right) = -\frac{5 g h L^2}{6}
$$

Repeating the same procedure for all the forces:

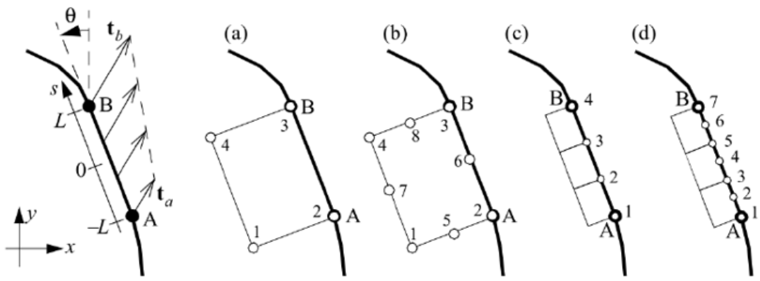
$$\boldsymbol{F}_b^{Analitycal} = -\frac{\rho g h L^2}{6}\begin{bmatrix}0\\5\\0\\5\\0\\7\\0\\7\end{bmatrix}, \boldsymbol{F}_b^{Gauss\ 1x1} = -\frac{\rho g h L^2}{6}\begin{bmatrix}0\\6\\0\\6\\0\\6\\0\\6\end{bmatrix}, \boldsymbol{F}_b^{Gauss\ 2x2} = -\frac{\rho g h L^2}{6}\begin{bmatrix}0\\5\\0\\5\\0\\7\\0\\7\end{bmatrix}$$

NOTE ABOUT THE INTEGRATION ACCURACY:

In the tables, the accuracy $n = 2m - 1$ (with m the number of integration points) is the order of the polynomial expression which can be integrated correctly with the Gauss approximation.

In this case $f(\xi,\eta)$ is a polynomial of order 2 in η and of order 1 in ξ. We need at least n=2 for η and n=1 for ξ. It means that, in order to have a correct integral, we need at least m=2 in η (which gives accuracy 3>2) and m=1 in ξ (which gives accuracy 1).

Problem 6.9



$$\boldsymbol{t} = \tfrac{1}{2}\left(1 - \tfrac{s}{L}\right)\boldsymbol{t_A} + \tfrac{1}{2}\left(1 + \tfrac{s}{L}\right)\boldsymbol{t_B}$$

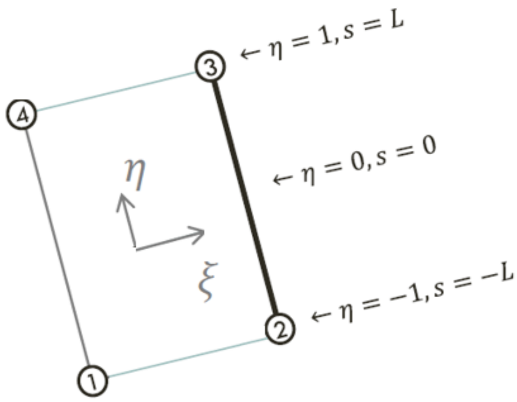
(a) 4 nodes elements

$$\boldsymbol{F_s} = \int_{S_e} \boldsymbol{N}^T \boldsymbol{t} dS$$

Change of Coordinates

$$s = L \cdot \eta \Leftrightarrow ds = L \cdot d\eta$$
$$dS = h \cdot ds \Leftrightarrow dS = h \cdot L \cdot d\eta$$

$$F_s = hL \int_{-1}^{1} \boldsymbol{N}^T \boldsymbol{t} d\eta$$



Load Function in Local Coordinates:

$$\boldsymbol{t}(s) = \frac{1}{2}\left(1 - \frac{s}{L}\right)\boldsymbol{t_A} + \frac{1}{2}\left(1 + \frac{s}{L}\right)\boldsymbol{t_B} \quad \Leftrightarrow \quad \boldsymbol{t}(\eta) = \frac{1}{2}\left(1 - \frac{\eta L}{L}\right)\boldsymbol{t_A} + \frac{1}{2}\left(1 + \frac{\eta L}{L}\right)\boldsymbol{t_B}$$

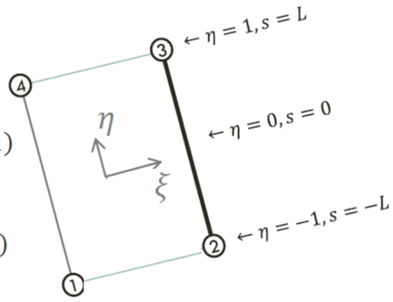
$$\boldsymbol{t_A} = \begin{bmatrix} t_{Ax} \\ t_{Ay} \end{bmatrix} \qquad \boldsymbol{t_B} = \begin{bmatrix} t_{Bx} \\ t_{By} \end{bmatrix} \qquad \boldsymbol{t}(\eta) = \frac{1}{2}\begin{bmatrix} (1-\eta)t_{Ax} + (1+\eta)t_{Bx} \\ (1-\eta)t_{Ay} + (1+\eta)t_{By} \end{bmatrix}$$

$$F_{i,x} = \frac{hL}{2} \int_{-1}^{1} N_i [(1-\eta)t_{Ax} + (1+\eta)t_{Bx}]\, \mathrm{d}\eta$$

$$F_{i,y} = \frac{hL}{2} \int_{-1}^{1} N_i [(1-\eta)t_{Ay} + (1+\eta)t_{By}]\, \mathrm{d}\eta$$

The load is along 2-3

$(\xi_1, \eta_1) = (-1,-1)$
$(\xi_2, \eta_2) = (1,-1)$
$(\xi_3, \eta_3) = (1,-1)$
$(\xi_4, \eta_4) = (-1,-1)$



Shape Functions:

$N_1 = (1-\xi)(1-\eta)/4 \quad N_2 = (1+\xi)(1-\eta)/4$
$N_3 = (1+\xi)(1+\eta)/4 \quad N_4 = (1-\xi)(1+\eta)/4$

Substitution:

$$F_{2,x} = \frac{hL}{2} \int_{-1}^{1} \left(\frac{(1+\xi)(1-\eta)}{4}\right) [(1-\eta)t_{Ax} + (1+\eta)t_{Bx}]\, \mathrm{d}\eta$$

$$F_{2,y} = \frac{hL}{2} \int_{-1}^{1} \left(\frac{(1+\xi)(1-\eta)}{4}\right) [(1-\eta)t_{Ay} + (1+\eta)t_{By}]\, \mathrm{d}\eta$$

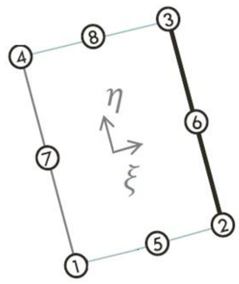
$$F_{3,x} = \frac{hL}{2} \int_{-1}^{1} \left(\frac{(1+\xi)(1+\eta)}{4}\right) [(1-\eta)t_{Ax} + (1+\eta)t_{Bx}]\, \mathrm{d}\eta$$

$$F_{3,y} = \frac{hL}{2} \int_{-1}^{1} \left(\frac{(1+\xi)(1+\eta)}{4}\right) [(1-\eta)t_{Ay} + (1+\eta)t_{By}]\, \mathrm{d}\eta$$

$\xi = 1$ in node 2 and node 3

$$\begin{aligned} F_{2,x} &= \frac{hL}{3}(2t_{Ax} + t_{Bx}) \\ F_{2,y} &= \frac{hL}{3}(2t_{Ay} + t_{By}) \\ F_{3,x} &= \frac{hL}{3}(t_{Ax} + 2t_{Bx}) \\ F_{3,y} &= \frac{hL}{3}(t_{Ay} + 2t_{By}) \end{aligned} \quad \Rightarrow \boldsymbol{F}_b = \frac{hL}{3}\begin{bmatrix} 0 \\ 0 \\ 2t_{Ax} + t_{Bx} \\ 2t_{Ay} + t_{By} \\ t_{Ax} + 2t_{Bx} \\ t_{Ay} + 2t_{By} \\ 0 \\ 0 \end{bmatrix}$$

(b) 8 nodes element



$(\xi_2, \eta_2) = (1, -1)$
$(\xi_3, \eta_3) = (1, 1)$
$(\xi_6, \eta_6) = (1, 0)$

$$N_i = \frac{(1+\xi_i\xi)(1+\eta_i\eta)(\xi_i\xi+\eta_i\eta-1)}{4} \qquad (i = 1,2,3,4)$$

$$N_i = \frac{(1-\xi^2)(1+\eta_i\eta)}{2} \qquad (i = 5,7)$$

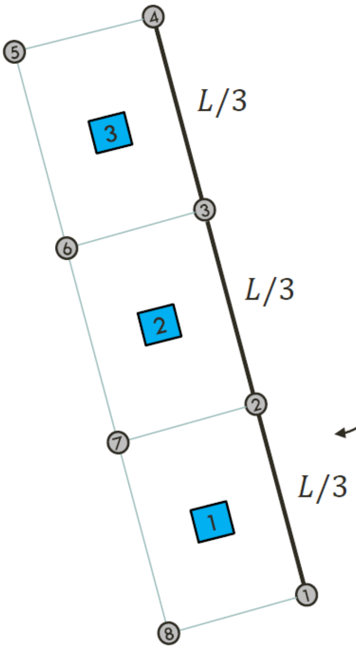
$$N_i = \frac{(1-\xi_i\xi)(1-\eta^2)}{2} \qquad (i = 6,8)$$

$$\boxed{\begin{aligned} N_1 &= -0.25(1-\xi)(1-\eta)(1+\xi+\eta) & N_5 &= 0.5(1-\xi^2)(1-\eta) \\ N_2 &= -0.25(1+\xi)(1-\eta)(1-\xi+\eta) & N_6 &= 0.5(1+\xi)(1-\eta^2) \\ N_3 &= -0.25(1+\xi)(1+\eta)(1-\xi-\eta) & N_7 &= 0.5(1-\xi^2)(1+\eta) \\ N_4 &= -0.25(1-\xi)(1+\eta)(1+\xi-\eta) & N_8 &= 0.5(1-\xi)(1-\eta^2) \end{aligned}}$$

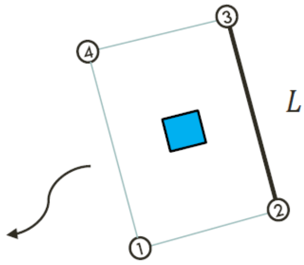
$$F_s = \int_{S_e} \mathbf{N}^{\mathsf{T}} \mathbf{t} dS = L \int_{z=0}^{z=h} \int_{\eta=-1}^{\eta=1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \\ N_5 & 0 \\ 0 & N_5 \\ N_6 & 0 \\ 0 & N_6 \\ N_7 & 0 \\ 0 & N_7 \\ N_8 & 0 \\ 0 & N_8 \end{bmatrix}_{\xi=1} \frac{1}{2} \begin{bmatrix} (1-\eta)\,\mathbf{t}_{Ax} + (1+\eta)\,\mathbf{t}_{Bx} \\ (1-\eta)\,\mathbf{t}_{Ay} + (1+\eta)\,\mathbf{t}_{By} \end{bmatrix} dz d\eta$$

$$
\boldsymbol{F}_s = \frac{hL}{3}\begin{bmatrix} 0 \\ 0 \\ t_{Ax} \\ t_{Ay} \\ t_{Bx} \\ t_{By} \\ 0 \\ 0 \\ 0 \\ 0 \\ 2(t_{Ax}+t_{Bx}) \\ 2(t_{Ay}+t_{By}) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

(c)



$$
\boldsymbol{t}_A = \boldsymbol{t}_B = \boldsymbol{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}
$$



LOCAL

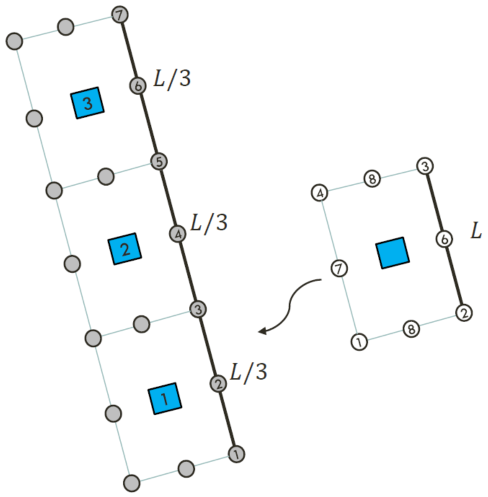
$$
\boldsymbol{F}_s = \frac{h\frac{L}{3}}{3}\begin{bmatrix} 0 \\ 0 \\ 2t_x + t_x \\ 2t_y + t_y \\ t_x + 2t_x \\ t_y + 2t_y \\ 0 \\ 0 \end{bmatrix}
\begin{array}{l} \text{1x} \\ \text{1y} \\ \text{2x} \\ \text{2y} \\ \text{3x} \\ \text{3y} \\ \text{4x} \\ \text{4y} \end{array}
= \frac{hL}{3}\begin{bmatrix} 0 \\ 0 \\ t_x \\ t_y \\ t_x \\ t_y \\ 0 \\ 0 \end{bmatrix}
$$

GLOBAL

$$
\boldsymbol{F}_{s,1} = \frac{hL}{3}\begin{bmatrix} 0 \\ 0 \\ t_x \\ t_y \\ t_x \\ t_y \\ 0 \\ 0 \end{bmatrix}
\begin{array}{l} F_{8x} \\ F_{8y} \\ F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{7x} \\ F_{7y} \end{array}
\qquad
\boldsymbol{F}_{s,2} = \frac{hL}{3}\begin{bmatrix} 0 \\ 0 \\ t_x \\ t_y \\ t_x \\ t_y \\ 0 \\ 0 \end{bmatrix}
\begin{array}{l} F_{7x} \\ F_{7y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{6x} \\ F_{6y} \end{array}
\qquad
\boldsymbol{F}_{s,3} = \frac{hL}{3}\begin{bmatrix} 0 \\ 0 \\ t_x \\ t_y \\ t_x \\ t_y \\ 0 \\ 0 \end{bmatrix}
\begin{array}{l} F_{6x} \\ F_{6y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \\ F_{5x} \\ F_{5y} \end{array}
$$

$$
\boldsymbol{F}_S^T = \frac{hL}{3}\begin{bmatrix} t_x & t_y & (t_x+t_x) & (t_y+t_y) & (t_x+t_x) & (t_y+t_y) & t_x & t_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

(d)



$$\boldsymbol{F}_{s,1} = \frac{h\left(\frac{L}{3}\right)}{3}\begin{bmatrix} 0 \\ 0 \\ t_x \\ t_y \\ t_x \\ t_y \\ 0 \\ 0 \\ 0 \\ 0 \\ 2(t_x + t_x) \\ 2(t_y + t_y) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{F}_{s,2} = \frac{h\left(\frac{L}{3}\right)}{3}\begin{bmatrix} 0 \\ 0 \\ t_x \\ t_y \\ t_x \\ t_y \\ 0 \\ 0 \\ 0 \\ 0 \\ 2(t_x + t_x) \\ 2(t_y + t_y) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{F}_{s,3} = \frac{h\left(\frac{L}{3}\right)}{3}\begin{bmatrix} 0 \\ 0 \\ t_x \\ t_y \\ t_x \\ t_y \\ 0 \\ 0 \\ 0 \\ 0 \\ 2(t_x + t_x) \\ 2(t_y + t_y) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_{1x} = \frac{hL}{9}t_x \quad F_{2x} = \frac{4hL}{9}t_x \quad F_{3x} = \frac{2hL}{9}t_x \quad F_{4x} = \frac{4hL}{9}t_x \quad F_{5x} = \frac{2hL}{9}t_x \quad F_{6x} = \frac{4hL}{9}t_x \quad F_{2x} = \frac{hL}{9}t_x$$

$$F_{1y} = \frac{hL}{9}t_y \quad F_{2y} = \frac{4hL}{9}t_y \quad F_{3y} = \frac{2hL}{9}t_y \quad F_{4y} = \frac{4hL}{9}t_y \quad F_{5y} = \frac{2hL}{9}t_y \quad F_{6y} = \frac{4hL}{9}t_y \quad F_{2y} = \frac{hL}{9}t_y$$