**Tutorial 4:** FEM for Engineering Applications (SE1025)

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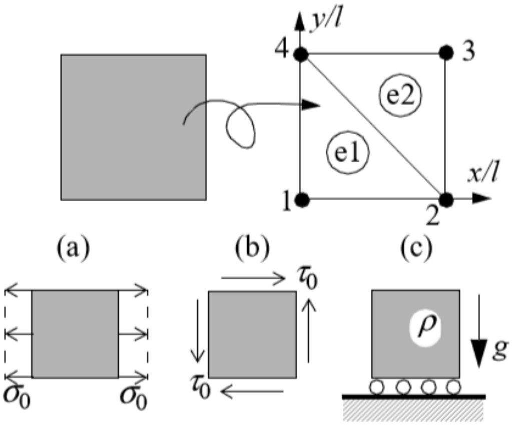
**6.14** Consider a thin quadratic sheet metal of size $l \times l$ and thickness $h$ of a linear elastic material ($E$, $\nu$). Model the sheet metal by use of two linear triangular elements (CST-element) and carry out FEM analyses for the three different load cases (a), (b) and (c). Introduce appropriate displacement boundary conditions, where symmetry conditions can be utilized, and determine node displacements and stresses in the elements. For simplicity, let Poisson's ratio be $\nu = 0$.




Load cases: (a) uniaxial tension,

(b) pure shear,

(c) dead weight, where $\rho$ is the density and $g$ acceleration of gravity.

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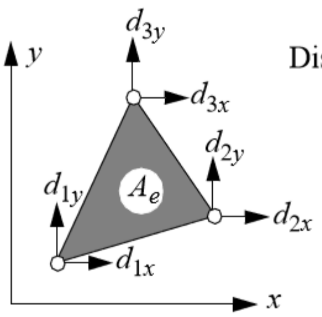
## Formulas

Plane (2D) triangular linear element:




Displacements:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \mathbf{d}_e = \mathbf{N}\mathbf{d}_e \qquad \mathbf{d}_e = \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}$$

$$N_1 = \frac{1}{2A_e}[(y_2 - y_3)(x - x_2) + (x_3 - x_2)(y - y_2)]$$

$$N_2 = \frac{1}{2A_e}[(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)]$$

$$N_3 = \frac{1}{2A_e}[(y_1 - y_2)(x - x_1) + (x_2 - x_1)(y - y_1)]$$

Strains:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \mathbf{B}\,\mathbf{d}_e \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \end{bmatrix} \qquad \mathbf{B}_i = \begin{bmatrix} \partial N_i/\partial x & 0 \\ 0 & \partial N_i/\partial y \\ \partial N_i/\partial y & \partial N_i/\partial x \end{bmatrix}$$

Stresses:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \qquad \text{(Plane stress)}$$

## Solution

### Theory Recall

The FEM equilibrium equation for the element is:

$$\int_{V_e} \mathbf{B}^\mathsf{T}\mathbf{C}\mathbf{B}dV\mathbf{d_e} = \int_{S_e} \mathbf{N}^\mathsf{T}\mathbf{t}dS + \int_{V_e} \mathbf{N}^\mathsf{T}\mathbf{K}dV$$

The displacement field is given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}$$

The shape functions are:

$$N_1(x,y) = \frac{1}{2A_e}[(y_2 - y_3)(x - x_2) + (x_3 - x_2)(y - y_2)]$$
$$N_2(x,y) = \frac{1}{2A_e}[(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)]$$
$$N_3(x,y) = \frac{1}{2A_e}[(y_1 - y_2)(x - x_1) + (x_2 - x_1)(y - y_1)]$$

In order to compute the Stiffness Matrix:

$$\int_{V_e} \mathbf{B}^\mathsf{T}\mathbf{C}\mathbf{B}dV = \mathbf{K_e}$$

we need :

**C**: the constitutive matrix for isotropic material in plane stress condition

$$\mathbf{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

**B**: the derivative of the shape function matrix

$$\mathbf{B} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} \end{bmatrix}$$

$$
\mathbf{B} = \frac{1}{2A_e}\begin{bmatrix} (y_2 - y_3) & 0 & (y_3 - y_1) & 0 & (y_1 - y_2) & 0 \\ 0 & (x_3 - x_2) & 0 & (x_1 - x_3) & 0 & (x_2 - x_1) \\ (x_3 - x_2) & (y_2 - y_3) & (x_1 - x_3) & (y_3 - y_1) & (x_2 - x_1) & (y_1 - y_2) \end{bmatrix}
$$

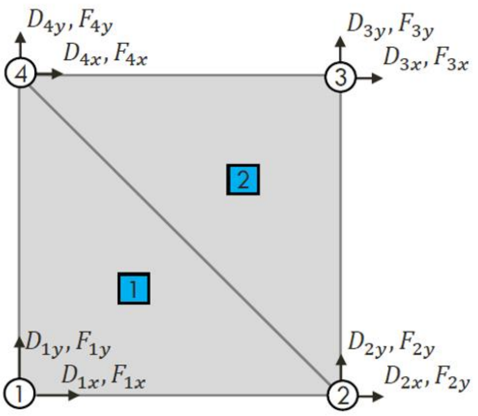
## Our Problem:

<u>*2 st CST-element*</u>

$$\boldsymbol{D}^T = [D_{1x} \quad D_{1y} \quad D_{2x} \quad D_{2y} \quad D_{3x} \quad D_{3y} \quad D_{4x} \quad D_{4y}]$$

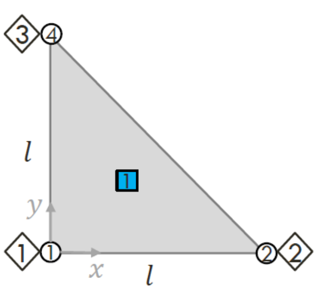
$$\boldsymbol{F}^T = [F_{1x} \quad F_{1y} \quad F_{2x} \quad F_{2y} \quad F_{3x} \quad F_{3y} \quad F_{4x} \quad F_{4y}]$$

$$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} dV \Rightarrow \boldsymbol{k}_e = \int_{A_e} h \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} dA$$




We consider the elements:

**Element 1**




Local Nodal Coordinates

$(x_1, y_1) = (0,0)$

$(x_2, y_2) = (l, 0)$

$(x_3, y_3) = (0, l)$

Area of the element

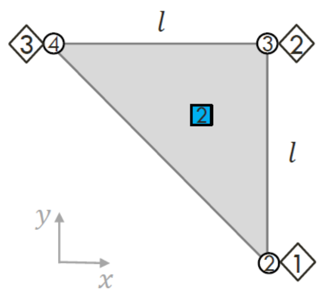
$$A_{e,1} = \frac{l \cdot l}{2} = \frac{l^2}{2}$$

$$N_1 = \frac{1}{2A_{e,1}}[(0-l)(x-l) + (0-l)(y-0)] = 1 - \frac{x}{l} - \frac{y}{l}$$

$$N_2 = \frac{1}{2A_{e,1}}[(l-0)(x-0) + (0-0)(y-l)] = \frac{x}{l}$$

$$N_3 = \frac{1}{2A_{e,1}}[(0-0)(x-0) + (l-0)(y-0)] = \frac{y}{l}$$

**Element 2**




Lokala nodernas koordinater

$(x_1, y_1) = (l, 0)$

$(x_2, y_2) = (l, l)$

$(x_3, y_3) = (0, l)$

Area of the element

$$A_{e,2} = \frac{l \cdot l}{2} = \frac{l^2}{2}$$

$$N_1 = \frac{1}{2A_{e,2}}[(l-l)(x-l) + (0-l)(y-l)] = 1 - \frac{y}{l}$$

$$N_2 = \frac{1}{2A_{e,2}}[(l-0)(x-0) + (l-0)(y-l)] = \frac{x}{l} + \frac{y}{l} - 1$$

$$N_3 = \frac{1}{2A_{e,2}}[(0-l)(x-l) + (l-l)(y-0)] = 1 - \frac{x}{l}$$

◇ Local node ○ Global node

Constitutive behavior:

$$\boldsymbol{C} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \Rightarrow \boldsymbol{C} = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \boxed{v = 0}$$

Element 1:

$$\boldsymbol{k}_{e1} = \int_{A_e} h\boldsymbol{B}_{e1}^T\boldsymbol{C}\boldsymbol{B}_{e1}dA = h\int_{A_e} \underbrace{\boldsymbol{B}_{e1}^T\boldsymbol{C}\boldsymbol{B}_{e1}}_{\text{constant}} dA = A_1 h\boldsymbol{B}_{e1}^T\boldsymbol{C}\boldsymbol{B}_{e1} = \frac{l^2}{2}h\boldsymbol{B}_{e1}^T\boldsymbol{C}\boldsymbol{B}_{e1}$$

$$\Rightarrow \boldsymbol{B}_{e1} = \frac{l}{2Ae,1}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} = \frac{1}{l}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{k}_{e1} = \frac{Eh}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Element 2:

$$\boldsymbol{k}_{e2} = \int_{A_e} h\boldsymbol{B}_{e2}^T\boldsymbol{C}\boldsymbol{B}_{e2}dA = h\int_{A_e} \underbrace{\boldsymbol{B}_{e2}^T\boldsymbol{C}\boldsymbol{B}_{e2}}_{\text{constant}} dA = A_2 h\boldsymbol{B}_{e2}^T\boldsymbol{C}\boldsymbol{B}_{e2} = \frac{l^2}{2}h\boldsymbol{B}_{e2}^T\boldsymbol{C}\boldsymbol{B}_{e2}$$

$$\Rightarrow \boldsymbol{B}_{e2} = \frac{l}{2Ae,2}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} = \frac{1}{l}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\boldsymbol{k}_{e2} = \frac{Eh}{4}\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 3 & 1 & -2 & -1 \\ -1 & -2 & 1 & 3 & 0 & -1 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

## Assembling:

$$\begin{array}{c} \\ \boldsymbol{k}_{e1} = \dfrac{Eh}{4} \end{array} \begin{array}{c} \begin{array}{cccccc} 1x & 1y & 2x & 2y & 4x & 4y \end{array} \\ \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1x \\ 1y \\ 2x \\ 2y \\ 4x \\ 4y \end{array}$$

$$\begin{array}{c} \\ \boldsymbol{k}_{e2} = \dfrac{Eh}{4} \end{array} \begin{array}{c} \begin{array}{cccccc} 2x & 2y & 3x & 3y & 4x & 4y \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 3 & 1 & -2 & -1 \\ -1 & -2 & 1 & 3 & 0 & -1 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 2x \\ 2y \\ 3x \\ 3y \\ 4x \\ 4y \end{array}$$

$$\Rightarrow \boldsymbol{K} = \frac{Eh}{4} \begin{array}{c} \begin{array}{cccccccc} 1x & 1y & 2x & 2y & 3x & 3y & 4x & 4y \end{array} \\ \begin{bmatrix} +3 & +1 & -2 & -1 & 0 & 0 & -1 & 0 \\ +1 & +3 & 0 & -1 & 0 & 0 & -1 & -2 \\ -2 & 0 & +2+1 & 0 & -1 & -1 & 0 & +1 \\ -1 & -1 & 0 & +1+2 & 0 & -2 & +1 & 0 \\ 0 & 0 & -1 & 0 & +3 & +1 & -2 & -1 \\ 0 & 0 & -1 & -2 & +1 & +3 & 0 & -1 \\ -1 & -1 & 0 & +1 & -2 & 0 & +1+2 & 0 \\ 0 & -2 & +1+1 & 0 & -1 & -1 & 0 & +2+1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \\ 4x \\ 4y \end{array}$$

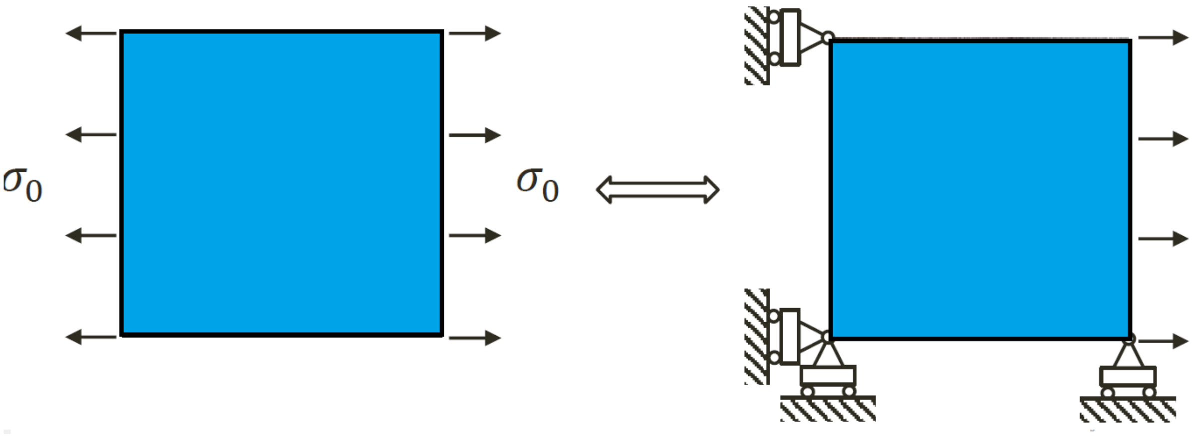
$$\Leftrightarrow \boldsymbol{K} = \frac{Eh}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & 0 & 0 & -1 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 & -1 & -2 \\ -2 & 0 & 3 & 0 & -1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 & -2 & -1 \\ 0 & 0 & -1 & -2 & 1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & -2 & 1 & 0 & -1 & -1 & 0 & 3 \end{bmatrix}$$

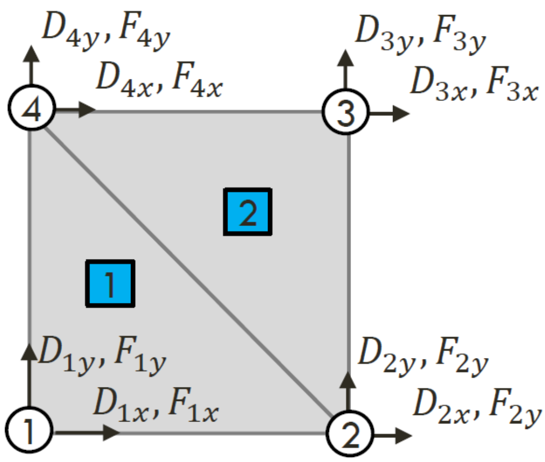
## Vector of forces

The expression of the element force vector for the CST comes from

$$\left[\int_{V_e} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} dV\right] \boldsymbol{d}_e = \underbrace{\left[\int_{S_e} \boldsymbol{N}^T \boldsymbol{t} dS + \int_{V_e} \boldsymbol{N}^T \boldsymbol{K}_{\boldsymbol{b}} dV\right]}_{\boldsymbol{F}_e}$$

LOAD CASE 1:




$$\boldsymbol{D} = \begin{bmatrix} D_{1x} \\ D_{1y} \\ D_{2x} \\ D_{2y} \\ D_{3x} \\ D_{3y} \\ D_{4x} \\ D_{4y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ D_{2x} \\ 0 \\ D_{3x} \\ D_{3y} \\ 0 \\ D_{4y} \end{bmatrix}$$

Vector of Nodal Forces:

$$\Longrightarrow \boldsymbol{F}_{nod}^{T} = [R_{1x} \quad R_{1y} \quad 0 \quad R_{2y} \quad 0 \quad 0 \quad R_{4x} \quad 0]$$

Vector of Superficial Forces:

(they act only on Element 2)

$$\boldsymbol{F}_{s,2} = \int_{S_e} \boldsymbol{N}^T \boldsymbol{t} dS = \int_{S_e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix} \underset{\Downarrow \; h\cdot dy}{dS} = \int_0^l \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ 0 \end{bmatrix} \cdot h \cdot dy = h \int_0^l \begin{bmatrix} N_1\sigma_0 \\ 0 \\ N_2\sigma_0 \\ 0 \\ N_3\sigma_0 \\ 0 \end{bmatrix} dy = h\sigma_0 \int_0^l \begin{bmatrix} 1 - \frac{y}{l} \\ 0 \\ \frac{x}{l} + \frac{y}{l} - 1 \\ 0 \\ 1 - \frac{x}{l} \\ 0 \end{bmatrix} dy$$

$$\Longrightarrow \boldsymbol{F}_{s,2}^{T} = \frac{\sigma_0 h l}{2} [1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$$

No Body Forces

$$\boldsymbol{F} = \boldsymbol{F}_{nod} + \boldsymbol{F}_s = \begin{bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ R_{2y} \\ 0 \\ 0 \\ R_{4x} \\ 0 \end{bmatrix} + \frac{\sigma_0 hl}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{1x} \\ R_{1y} \\ \frac{\sigma_0 hl}{2} \\ R_{2y} \\ \frac{\sigma_0 hl}{2} \\ 0 \\ R_{4x} \\ 0 \end{bmatrix}$$

Solving the system:

$$\boldsymbol{F} = \boldsymbol{KD} \qquad \Rightarrow \boldsymbol{F}_{red} = \boldsymbol{K}_{red}\boldsymbol{D}_{red}$$

$$\begin{bmatrix} R_{1x} \\ R_{1y} \\ \frac{\sigma_0 hl}{2} \\ R_{2y} \\ \frac{\sigma_0 hl}{2} \\ 0 \\ R_{4x} \\ 0 \end{bmatrix} = \frac{Eh}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & 0 & 0 & -1 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 & -1 & -2 \\ -2 & 0 & 3 & 0 & -1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 & -2 & -1 \\ 0 & 0 & -1 & -2 & 1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & -2 & 1 & 0 & -1 & -1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ D_{2x} \\ 0 \\ D_{3x} \\ D_{3y} \\ 0 \\ D_{4y} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sigma_0 hl}{2} \\ \frac{\sigma_0 hl}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{Eh}{4}\begin{bmatrix} 3 & -1 & -1 & 1 \\ -1 & 3 & 1 & -1 \\ -1 & 1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} D_{2x} \\ D_{3x} \\ D_{3y} \\ D_{4y} \end{bmatrix}$$

Displacements:

$$\Rightarrow \begin{bmatrix} D_{2x} \\ D_{3x} \\ D_{3y} \\ D_{4y} \end{bmatrix} = \frac{\sigma_0 l}{E}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Forces:

$$\begin{bmatrix} R_{1x} \\ R_{1y} \\ \frac{\sigma_0 hl}{2} \\ R_{2y} \\ \frac{\sigma_0 hl}{2} \\ 0 \\ R_{4x} \\ 0 \end{bmatrix} = \frac{Eh}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & 0 & 0 & -1 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 & -1 & -2 \\ -2 & 0 & 3 & 0 & -1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 & -2 & -1 \\ 0 & 0 & -1 & -2 & 1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & -2 & 1 & 0 & -1 & -1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \frac{\sigma_0 l}{E} \\ 0 \\ \frac{\sigma_0 l}{E} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} R_{1x} \\ R_{1y} \\ R_{2y} \\ R_{4x} \end{bmatrix} = \frac{\sigma_0 hl}{2}\begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

## Post- Processing

### Element 1

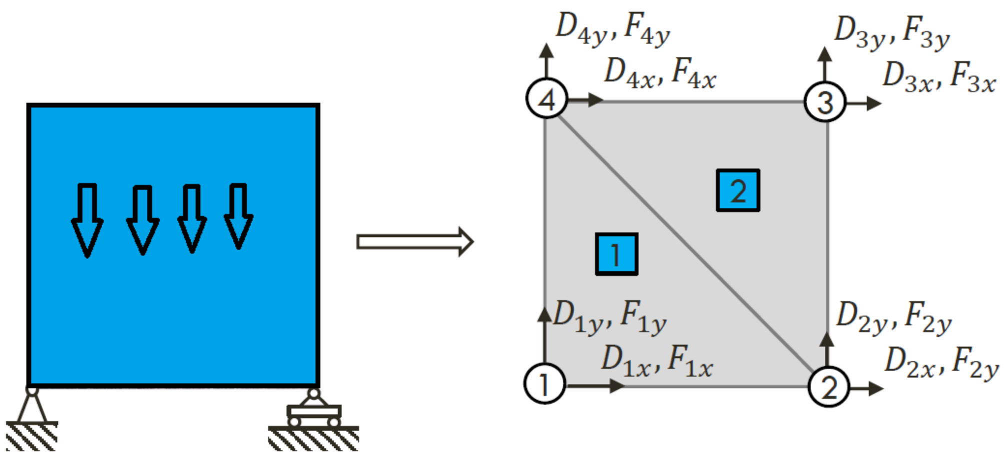
$$\boldsymbol{\sigma_1} = \underbrace{E\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}}_{\boldsymbol{C_1}|_{v=0}} \underbrace{\frac{1}{l}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 1\\ -1 & -1 & 0 & 1 & 1 & 0\end{bmatrix}}_{\boldsymbol{B_1}} \underbrace{\frac{\sigma_0 l}{E}\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}}_{\boldsymbol{d_{e,1}}}\begin{matrix}\text{D 1x}\\ \text{D 1y}\\ \text{D 2x}\\ \text{D 2y}\\ \text{D 4x}\\ \text{D 4y}\end{matrix} \Leftrightarrow \boldsymbol{\sigma_1} = \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}\sigma_0\\ 0\\ 0\end{bmatrix}$$

### Element 2

$$\boldsymbol{\sigma_2} = \underbrace{E\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}}_{\boldsymbol{C_2}|_{v=0}} \underbrace{\frac{1}{l}\begin{bmatrix}0 & 0 & 1 & 0 & -1 & 0\\ 0 & -1 & 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 1 & 1 & -1\end{bmatrix}}_{\boldsymbol{B_2}} \underbrace{\frac{\sigma_0 l}{E}\begin{bmatrix}1\\0\\1\\0\\0\\0\end{bmatrix}}_{\boldsymbol{d_{e,2}}}\begin{matrix}\text{D 2x}\\ \text{D 2y}\\ \text{D 3x}\\ \text{D 3Y}\\ \text{D 4x}\\ \text{D 4y}\end{matrix} \Leftrightarrow \boldsymbol{\sigma_2} = \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}\sigma_0\\ 0\\ 0\end{bmatrix}$$

We only have tension in the X direction. OK !

LOAD CASE 3:




### Force Vector

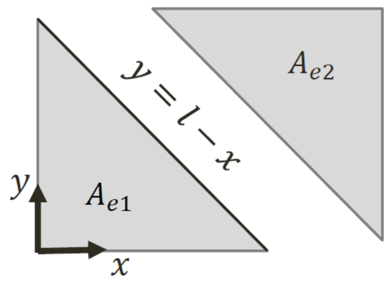
### Nodal Force Vector

$$\Longrightarrow \boldsymbol{F}_{nod}^{T} = [R_{1x} \quad R_{1y} \quad 0 \quad R_{2y} \quad 0 \quad 0 \quad 0 \quad 0]$$

## Body Force Vector:

$$\boldsymbol{F}_{b,e} = \int_{V_e} \boldsymbol{N}^T \boldsymbol{K}_b dV$$

$$\Rightarrow \boldsymbol{F}_{b,e} = \int_{V_e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} K_x \\ K_y \end{bmatrix} dV = \int_{A_e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} K_x \\ K_y \end{bmatrix} hdA = \iint \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} K_x \\ K_y \end{bmatrix} h \cdot dx \cdot dy$$




$$\Rightarrow \boldsymbol{F}_{b,1} = \int_0^l \int_0^{l-x} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} h \cdot dy \cdot dx = (\ldots)$$

$$\Rightarrow \boldsymbol{F}_{b,2} = \int_0^l \int_{l-x}^{l} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} h \cdot dy \cdot dx = (\ldots)$$

## Element 1:

$$\Rightarrow \boldsymbol{F}_{b,1} = \int_0^l \int_0^{l-x} -\rho g \begin{bmatrix} 0 \\ 1 - \frac{x}{l} - \frac{y}{l} \\ 0 \\ \frac{x}{l} \\ 0 \\ \frac{y}{l} \end{bmatrix} h \cdot dy \cdot dx = \frac{-\rho g h}{2l} \int_0^l \begin{bmatrix} 0 \\ l^2 - 2lx + x^2 \\ 0 \\ 2lx - 2x^2 \\ 0 \\ l^2 - 2lx + x^2 \end{bmatrix} dx = -\frac{\rho g h l^2}{6} \begin{array}{c} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \end{array} \begin{array}{c} \scriptstyle 1x \\ \scriptstyle 1y \\ \scriptstyle 2x \\ \scriptstyle 2y \\ \scriptstyle 4x \\ \scriptstyle 4y \end{array}$$

## Element 2:

$$\Rightarrow \boldsymbol{F}_{b,2} = \int_0^l \int_{l-x}^{l} -\rho g \begin{bmatrix} 0 \\ 1 - \frac{y}{l} \\ 0 \\ \frac{x}{l} + \frac{y}{l} - 1 \\ 0 \\ 1 - \frac{x}{l} \end{bmatrix} h \cdot dy \cdot dx = \; \cdots \; = -\frac{\rho g h l^2}{6} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{array}{c} \scriptstyle 2x \\ \scriptstyle 2y \\ \scriptstyle 3x \\ \scriptstyle 3y \\ \scriptstyle 4x \\ \scriptstyle 4y \end{array}$$

$$\boldsymbol{F} = \boldsymbol{F}_{nod} + \boldsymbol{F}_{b,1} + \boldsymbol{F}_{b,2} = \begin{bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ R_{2y} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{\rho g h l^2}{6} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\rho g h l^2}{6} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{1x} \\ R_{1y} - \frac{\rho g h l^2}{6} \\ 0 \\ R_{2y} - \frac{\rho g h l^2}{3} \\ 0 \\ -\frac{\rho g h l^2}{6} \\ 0 \\ -\frac{\rho g h l^2}{3} \end{bmatrix}$$

Solving the system.

$$\begin{bmatrix} R_{1x} \\ R_{1y} - \frac{\rho g h l^2}{6} \\ 0 \\ R_{2y} - \frac{\rho g h l^2}{3} \\ 0 \\ -\frac{\rho g h l^2}{6} \\ 0 \\ -\frac{\rho g h l^2}{3} \end{bmatrix} = \frac{Eh}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & 0 & 0 & -1 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 & -1 & -2 \\ -2 & 0 & 3 & 0 & -1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 & -2 & -1 \\ 0 & 0 & -1 & -2 & 1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & -2 & 1 & 0 & -1 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ D_{2x} \\ 0 \\ D_{3x} \\ D_{3y} \\ D_{4x} \\ D_{4y} \end{bmatrix}$$

$$\Rightarrow \boldsymbol{F}_{red} = \boldsymbol{K}_{red} \boldsymbol{D}_{red} \Leftrightarrow$$

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{\rho g h l^2}{6} \\ 0 \\ -\frac{\rho g h l^2}{3} \end{bmatrix} = \frac{Eh}{4} \begin{bmatrix} 3 & -1 & -1 & 0 & 1 \\ -1 & 3 & 1 & -2 & -1 \\ -1 & 1 & 3 & 0 & -1 \\ 0 & -2 & 0 & 3 & 0 \\ 1 & -1 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{3x} \\ D_{3y} \\ D_{4x} \\ D_{4y} \end{bmatrix} \Rightarrow \begin{bmatrix} D_{2x} \\ D_{3x} \\ D_{3y} \\ D_{4x} \\ D_{4y} \end{bmatrix} = \frac{\rho g l^2}{24E} \begin{bmatrix} 1 \\ -3 \\ -9 \\ -2 \\ -15 \end{bmatrix}$$

## Finding the reaction forces

$$\begin{bmatrix} R_{1x} \\ R_{1y} - \frac{\rho g h l^2}{6} \\ 0 \\ R_{2y} - \frac{\rho g h l^2}{3} \\ 0 \\ -\frac{\rho g h l^2}{6} \\ 0 \\ -\frac{\rho g h l^2}{3} \end{bmatrix} = \frac{Eh}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & 0 & 0 & -1 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 & -1 & -2 \\ -2 & 0 & 3 & 0 & -1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 & -2 & -1 \\ 0 & 0 & -1 & -2 & 1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & -2 & 1 & 0 & -1 & -1 & 0 & 3 \end{bmatrix} \frac{\rho g l^2}{24E} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -3 \\ -9 \\ -2 \\ -15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix} = \begin{bmatrix} R_{1x} \\ R_{1y} - \frac{\rho g h l^2}{6} \\ 0 \\ R_{2y} - \frac{\rho g h l^2}{3} \\ 0 \\ -\frac{\rho g h l^2}{6} \\ 0 \\ -\frac{\rho g h l^2}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\rho g h l^2}{3} \\ 0 \\ \frac{\rho g h l^2}{6} \\ 0 \\ \frac{\rho g h l^2}{6} \\ 0 \\ -\frac{\rho g h l^2}{6} \\ 0 \\ -\frac{\rho g h l^2}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} R_{1x} \\ R_{1y} \\ R_{2y} \end{bmatrix} = \begin{bmatrix} \frac{\rho g h l^2}{2} \\ \frac{\rho g h l^2}{3} \\ \frac{\rho g h l^2}{2} \end{bmatrix}$$

## Post-Processing

$$\sigma = CBD_e$$

$$\boldsymbol{\sigma_1} = \underbrace{E\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}}_{\boldsymbol{C}|_{v=0}} \underbrace{\frac{1}{l}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 1\\ -1 & -1 & 1 & 1 & 1 & 0\end{bmatrix}}_{\boldsymbol{B}_1} \underbrace{\frac{\rho g l^2}{24E}\begin{bmatrix}0\\0\\1\\0\\-2\\-15\end{bmatrix}}_{\boldsymbol{d}_{e,1}} \Leftrightarrow \boldsymbol{\sigma_1} = \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \frac{\rho g l^2}{24}\begin{bmatrix}1\\-15\\-1\end{bmatrix}$$

$$\boldsymbol{\sigma_2} = \underbrace{E\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}}_{\boldsymbol{C}|_{v=0}} \underbrace{\frac{1}{l}\begin{bmatrix}0 & 0 & 1 & 0 & -1 & 0\\ 0 & -1 & 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 1 & 0 & -1\end{bmatrix}}_{\boldsymbol{B}_2} \underbrace{\frac{\rho g l^2}{24E}\begin{bmatrix}1\\0\\-3\\-9\\-2\\-15\end{bmatrix}}_{\boldsymbol{d}_{e,2}} \Leftrightarrow \boldsymbol{\sigma_1} = \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \frac{\rho g l^2}{24}\begin{bmatrix}-1\\-9\\1\end{bmatrix}$$