Tutorial 2: FEM for Engineering Applications (SE1025)

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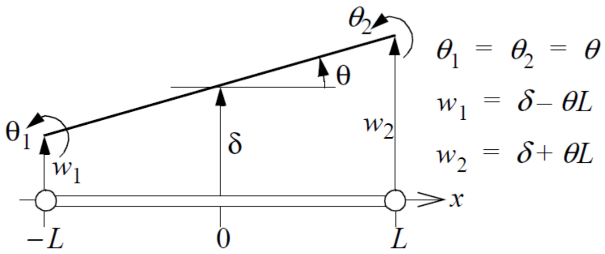
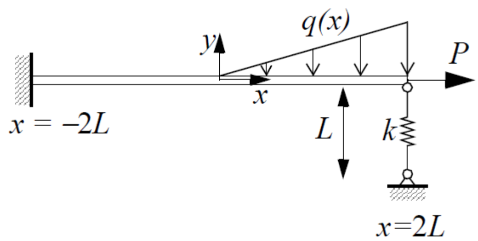
4.6 One requirement the displacement interpolation in an element must satisfy, is that it should be able to model an arbitrary rigid body motion. For a plane 2-node beam element with 2 degrees of freedom at each node, this means that the deflection of the beam must be able to take the form



$$w(x) = \delta + \theta x,$$

where δ and θ are parameters describing an arbitrary rigid body motion as illustrated in the figure to the right. Show that the displacement interpolation of the element can satisfy a rigid body motion as described above.

5.4 A force per unit length $q(x)$ is applied on a beam with elastic modulus E and a cross sectional area A and a moment of inertia I. The left end of the beam is clamped and the right end rests on an elastic support, here modelled by a vertical spring with spring constant $k = \eta EI/L^3$.



(a) Carry out a finite element analysis, where the beam is modelled by one two-node element, and evaluate the deflection of the beam. Here: $\eta = 3/2$ and $q(x) = -Q/(2L)(x/L)$.

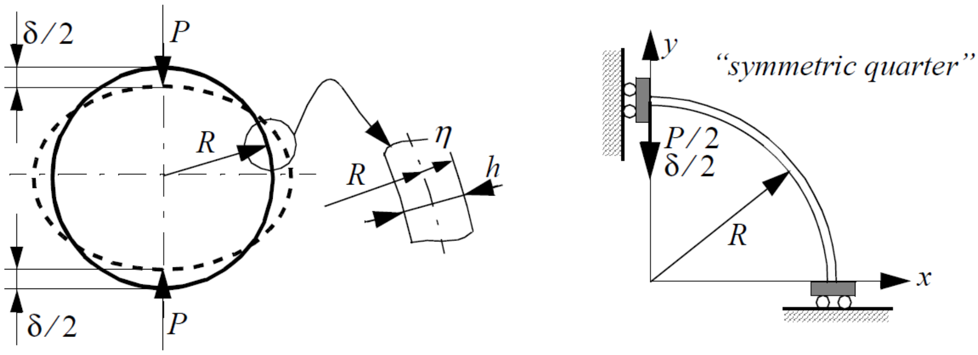
(b) Divide the beam into two element of equal to length and redo the analysis.

Note that the deflection at the nodes will in the current case always coincide with the exact solution. The deflection between the nodes for $0 \le x \le 2L$ will deviate somewhat from the exact solution due to distributed load $q(x)$.

5.6 The figure below shows a circular ring, which is an integral part of a flexible machine member. The ring is subjected to diametrically opposed forces according to the figure. Determine the spring constant defined as $k = P/\delta$ by use of FEM. If the symmetry of the problem is fully utilized, only a quarter of the ring needs to be modelled. The problem can for instance be analysed by the Matlab based FEM program "frame2D", available at the home page of the course. If the displacement, δ, primarily is due to bending deformation (a good approximation if $R \gg h$), the spring constant of the ring can analytically be expressed as

$$k = \frac{4\pi}{(\pi^2 - 8)}\frac{EI}{R^3},$$

where E is the elastic modulus and I area moment of inertia. Note that in order for the FEM solution to come close to this result, the FEM model requires that $R \gg h$.



Problem 4.6

Displacement field for a beam element:

$$w(x) = \boldsymbol{N}\boldsymbol{d}_e = \underbrace{[N_1 \quad N_2 \quad N_3 \quad N_4]}_{\boldsymbol{N}} \underbrace{\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}}_{\boldsymbol{d}_e} \Rightarrow w(x) = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{bmatrix} \delta - \theta L \\ \theta \\ \delta + \theta L \\ \theta \end{bmatrix}$$

Shape Functions

$$N_1 = \frac{2-3\xi+\xi^3}{4} \qquad N_2 = \frac{L(1-\xi-\xi^2+\xi^3)}{4}$$

$$N_3 = \frac{2+3\xi-\xi^3}{4} \qquad N_4 = \frac{L(-1-\xi+\xi^2+\xi^3)}{4}$$

Some manipulations:

$$\Rightarrow w(\xi) = N_1(\delta - \theta L) + N_2\theta + N_3(\delta + \theta L) + N_4\theta$$

$$\Leftrightarrow w(\xi) = \delta(N_1 + N_3) + \theta(N_2 + N_4) + \theta L(N_3 - N_1)$$

$$\boxed{w(\xi) = \delta\left(\frac{2-3\xi+\xi^3}{4} + \frac{2+3\xi-\xi^3}{4}\right) + \theta\left(\frac{L(1-\xi-\xi^2+\xi^3)}{4} + \frac{L(-1-\xi+\xi^2+\xi^3)}{4}\right) + \theta L\left(\frac{2+3\xi-\xi^3}{4} - \frac{2-3\xi+\xi^3}{4}\right)}$$

$$4 \cdot w(\xi) = \delta(2-3\xi+\xi^3+2+3\xi-\xi^3) + \theta\big(L(1-\xi-\xi^2+\xi^3) + L(-1-\xi+\xi^2+\xi^3)\big) + \theta L(2+3\xi-\xi^3-2+3\xi-\xi^3)$$

$$4w(\xi) = \delta(2-3\xi+\xi^3+2+3\xi-\xi^3) + \theta L(1-\xi-\xi^2+\xi^3-1-\xi+\xi^2+\xi^3+2+3\xi-\xi^3-2+3\xi-\xi^3)$$

$$4w(\xi) = \delta(4+0+0) + \theta L(0+4\xi+0+0)$$

$$4w(\xi) = \delta(4) + \theta L(4\xi)$$

$$w(\xi) = \delta + \theta L\xi$$

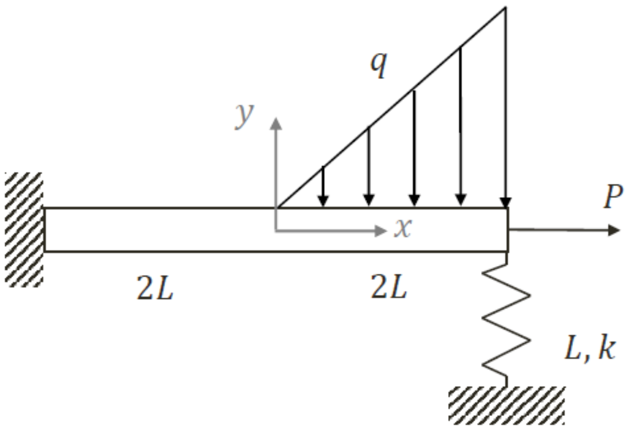
Changing the coordinate system:

$$0 \le \xi \le 1 \Leftrightarrow 0 \le x \le L \Rightarrow x = L\xi \Leftrightarrow \xi = \frac{x}{L}$$

$$w(x) = \delta + \theta L\frac{x}{L} \Rightarrow w(x) = \delta + \theta x$$

OK!

Problem 5.4



Data:

$E \quad A \quad I$

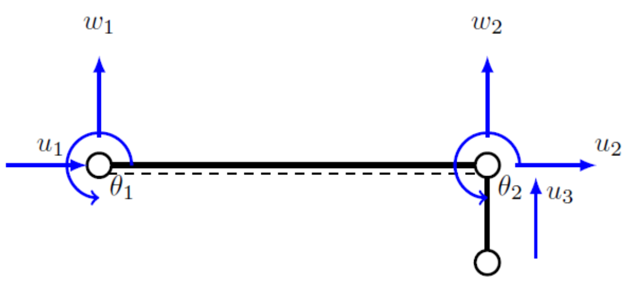
$$k = \frac{3EI}{2L^3}$$

$$q(x) = -\frac{Q}{2L}\left(\frac{x}{L}\right)$$

Find the deflection

Beam Element:

It is loaded with both axial and bending forces → Truss Model + Beam Model



Pure Beam (Eulero-Bernulli) Equations (recall):

$$\boxed{\frac{d^2M}{dx^2} + q = 0}$$

$$\int_a^b v \frac{d^2M}{dx^2} dx + \int_a^b v q dx = 0$$

v : weigth function
Integration over the domain

$$\left[v\frac{dM}{dx}\right]_a^b - \int_a^b \frac{dv}{dx}\frac{dM}{dx}dx + \int_a^b vqdx = 0$$

integration by parts

$$\left[v\frac{dM}{dx}\right]_a^b - \left[\frac{dv}{dx}M\right]_a^b + \int_a^b \frac{d^2v}{dx^2}M dx + \int_a^b vqdx = 0$$

integration by parts

Shape functions interpolation

$$w(x) = \mathbf{N}\mathbf{d} \qquad \mathbf{B} = \frac{d^2\mathbf{N}}{dx^2}$$

Galerkin's Method

$$v = \mathbf{N}\mathbf{c} = \mathbf{c}^{\mathsf{T}}\mathbf{N}^{\mathsf{T}} \qquad \frac{dv}{dx} = \mathbf{c}^{\mathsf{T}}\frac{d\mathbf{N}^{\mathsf{T}}}{dx} \qquad \frac{d^2v}{dx^2} = \mathbf{c}^{\mathsf{T}}\frac{d^2\mathbf{N}^{\mathsf{T}}}{dx^2} = \mathbf{c}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}$$

We substitute the interpolation of v(x) and w(x) in the weak form of the equilibrium equation:

$$\int_a^b vq\,dx + \left[v\frac{dM}{dx}\right]_a^b - \left[\frac{dv}{dx}M\right]_a^b + \int_a^b \frac{d^2v}{dx^2}M\,dx = 0$$

$$\Downarrow$$

$$\mathbf{c}^{\mathsf{T}}\left(\int_a^b \mathbf{N}^{\mathsf{T}}q\,dx + \left[\mathbf{N}^{\mathsf{T}}\frac{dM}{dx}\right]_a^b - \left[\frac{d\mathbf{N}^{\mathsf{T}}}{dx}M\right]_a^o + \int_a^b \mathbf{B}^{\mathsf{T}}M\,dx\right) = 0$$

We write the moment M as a function of the curvature:

$$M = -EI\frac{d^2w}{dx^2}$$
$$M = -EI\mathbf{B}\mathbf{d}$$

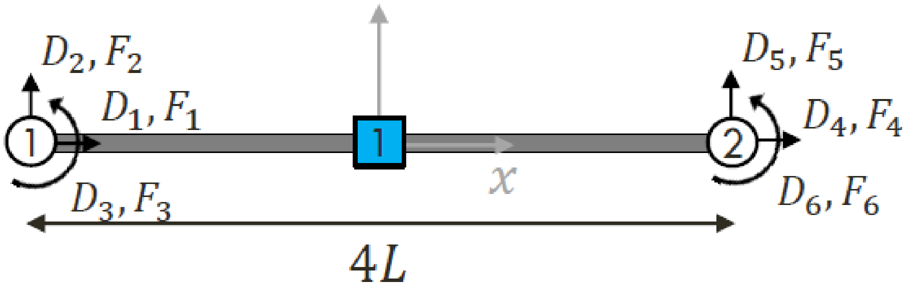
And we get:

$$\int_a^b \mathbf{N}^{\mathsf{T}}q\,dx + \left[\mathbf{N}^{\mathsf{T}}\frac{dM}{dx}\right]_a^b - \left[\frac{d\mathbf{N}^{\mathsf{T}}}{dx}M\right]_a^b = \int_a^b \mathbf{B}^{\mathsf{T}}EI\mathbf{B}\mathbf{d}\,dx$$

Distributed Load Nodal Shear Nodal Moment STIFFNESS MATRIX

$$\int_a^b \mathbf{N}^{\mathsf{T}}q\,dx + \left[\mathbf{N}^{\mathsf{T}}\frac{dM}{dx}\right]_a^b - \left[\frac{d\mathbf{N}^{\mathsf{T}}}{dx}M\right]_a^b = \int_a^b \mathbf{B}^{\mathsf{T}}EI\mathbf{B}\,dx\;\mathbf{d}$$

1) Stiffness Matrix of Element 1 : BEAM



Axial Component (Truss)

$$\boldsymbol{k}_{e1}^{\text{Truss}} = \frac{EA}{4L}\begin{array}{c} \begin{array}{cc} D_1 & D_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} D_1 \\ D_4 \end{array}$$

Bending Component (Pure Beam)

General Formulation:

$$\boldsymbol{k}_{e,i} = \frac{EI}{2a^3}\begin{bmatrix} 3 & 3a & -3 & 3a \\ 3a & 4a^2 & -3a & 2a^2 \\ -3 & -3a & 3 & -3a \\ 3a & 2a^2 & -3a & 4a^2 \end{bmatrix} \qquad L_e = 2a = 2L$$

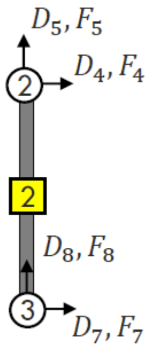
In our case a = 2L

$$\boldsymbol{k}_{e,1}^{\text{Beam}} = \frac{EI}{16L^3}\begin{array}{c} \begin{array}{cccc} D_2 & D_3 & D_5 & D_6 \end{array} \\ \begin{bmatrix} 3 & 6L & -3 & 6L \\ 6L & 16L^2 & -6L & 16L^2 \\ -3 & -6L & 3 & -6L \\ 6L & 8L^2 & -6L & 8L^2 \end{bmatrix} \end{array}\begin{array}{c} D_2 \\ D_3 \\ D_5 \\ D_6 \end{array}$$

$$\Rightarrow \boldsymbol{k}_{e1} = \begin{array}{c} \begin{array}{cccccc} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \end{array} \\ \begin{bmatrix} \frac{EA}{4L} & 0 & 0 & -\frac{EA}{4L} & 0 & 0 \\ 0 & 3\frac{EI}{16L^3} & 6L\frac{EI}{16L^3} & 0 & -3\frac{EI}{16L^3} & 6L\frac{EI}{16L^3} \\ 0 & 6L\frac{EI}{16L^3} & 16L^2\frac{EI}{16L^3} & 0 & -6L\frac{EI}{16L^3} & 8L^2\frac{EI}{16L^3} \\ -\frac{EA}{4L} & 0 & 0 & \frac{EA}{4L} & 0 & 0 \\ 0 & -3\frac{EI}{16L^3} & -6L\frac{EI}{16L^3} & 0 & 3\frac{EI}{16L^3} & -6L\frac{EI}{16L^3} \\ 0 & 6L\frac{EI}{16L^3} & 8L^2\frac{EI}{16L^3} & 0 & -6L\frac{EI}{16L^3} & 16L^2\frac{EI}{16L^3} \end{bmatrix} \end{array}\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{array}$$

$$\boldsymbol{k}_{e1} = \begin{array}{cc} \begin{matrix} D_1 & \quad D_2 & \quad D_3 & \quad D_4 & \quad D_5 & \quad D_6 \end{matrix} & \\ \begin{bmatrix} \frac{EA}{4L} & 0 & 0 & -\frac{EA}{4L} & 0 & 0 \\ 0 & \frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & -\frac{3EI}{16L^3} & \frac{3EI}{8L^2} \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{2L} \\ -\frac{EA}{4L} & 0 & 0 & \frac{EA}{4L} & 0 & 0 \\ 0 & -\frac{3EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & \frac{3EI}{16L^3} & -\frac{3EI}{8L^2} \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{L} \end{bmatrix} & \begin{matrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{matrix} \end{array}$$

2) Stiffness Matrix of Element 2 : Spring



$$\boldsymbol{a} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{k}_{e,2} = k \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \frac{3EI}{2L^3} \begin{array}{cc} \begin{matrix} D_4 & D_5 & D_7 & D_8 \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} & \begin{matrix} D_4 \\ D_5 \\ D_7 \\ D_8 \end{matrix} \end{array}$$

3) Assemble the Stiffness Matrix

$$\boldsymbol{k}_{e1} = \begin{array}{cc} \begin{matrix} D_1 & \quad D_2 & \quad D_3 & \quad D_4 & \quad D_5 & \quad D_6 \end{matrix} & \\ \begin{bmatrix} \frac{EA}{4L} & 0 & 0 & -\frac{EA}{4L} & 0 & 0 \\ 0 & \frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & -\frac{3EI}{16L^3} & \frac{3EI}{8L^2} \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{2L} \\ -\frac{EA}{4L} & 0 & 0 & \frac{EA}{4L} & 0 & 0 \\ 0 & -\frac{3EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & \frac{3EI}{16L^3} & -\frac{3EI}{8L^2} \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{L} \end{bmatrix} & \begin{matrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{matrix} \end{array}$$

$$\boldsymbol{k}_{e,2} = \frac{3EI}{2L^3} \begin{array}{cc} \begin{matrix} D_4 & D_5 & D_7 & D_8 \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} & \begin{matrix} D_4 \\ D_5 \\ D_7 \\ D_8 \end{matrix} \end{array}$$

DOF	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
D_1	$\frac{EA}{4L}$			$-\frac{EA}{4L}$				
D_2		$\frac{3EI}{16L^3}$	$\frac{3EI}{8L^2}$		$-\frac{3EI}{16L^3}$	$\frac{3EI}{8L^2}$		
D_3		$\frac{3EI}{8L^2}$	$\frac{EI}{L}$		$-\frac{3EI}{8L^2}$	$\frac{EI}{2L}$		
D_4	$-\frac{EA}{4L}$			$\frac{EA}{4L}$				
D_5		$-\frac{3EI}{16L^3}$	$-\frac{3EI}{8L^2}$		$\frac{3EI}{16L^3}+\frac{3EI}{2L^3}$	$-\frac{3EI}{8L^2}$		$-\frac{3EI}{2L^3}$
D_6		$\frac{3EI}{8L^2}$	$\frac{EI}{2L}$		$-\frac{3EI}{8L^2}$	$\frac{EI}{L}$		
D_7								
D_8					$-\frac{3EI}{2L^3}$			$\frac{3EI}{2L^3}$

$$
\boldsymbol{K} = \begin{bmatrix}
\frac{EA}{4L} & 0 & 0 & -\frac{EA}{4L} & 0 & 0 & 0 & 0 \\
0 & \frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & -\frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & 0 \\
0 & \frac{3EI}{8L^2} & \frac{EI}{L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & 0 \\
-\frac{EA}{4L} & 0 & 0 & \frac{EA}{4L} & 0 & 0 & 0 & 0 \\
0 & -\frac{3EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & \frac{27EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & -\frac{3EI}{2L^3} \\
0 & \frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3EI}{2L^3} & 0 & 0 & \frac{3EI}{2L^3}
\end{bmatrix}
$$

4) Define the Force Vector

$$\boldsymbol{F} = \boldsymbol{F}_b + \boldsymbol{F}_s$$

$$
\boldsymbol{F} = \begin{bmatrix} F_1 \\ F_2 \\ M_3 \\ F_4 \\ F_5 \\ M_6 \\ F_7 \\ F_8 \end{bmatrix} = \boldsymbol{F}_{b,1} + \boldsymbol{F}_{b,2} + \boldsymbol{F}_s \qquad \boldsymbol{F}_s = \begin{bmatrix} R_1 \\ R_2 \\ M_{R3} \\ P \\ 0 \\ 0 \\ 0 \\ R_8 \end{bmatrix}
$$

5) Compute the body force vector for element 1

In general, we express the body force through the shape functions

$$\boldsymbol{F}_b = \int \boldsymbol{N}^T q(x) dx$$

For Element 1 we can write

$$\boldsymbol{F}_{b,1} = \int_{-2L}^{2L} \boldsymbol{N}^T q(x) dx$$

Shape Functions:

$$
\boldsymbol{N}^T = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} (2 - 3\xi + \xi^3)/4 \\ L(1 - \xi - \xi^2 + \xi^3)/4 \\ (2 + 3\xi - \xi^3)/4 \\ L(-1 - \xi + \xi^2 + \xi^3)/4 \end{bmatrix}
$$

Body Force:

$$
q(x) = \begin{cases} 0\,, -2L \le x < 0 \\ -\frac{Q}{2L} \cdot \frac{x}{L}\,, 0 \le x \le 2L \end{cases}
$$

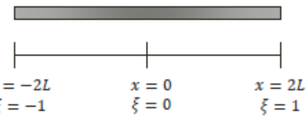
$$
\boldsymbol{F}_{b,1} = \int_{-2L}^{2L} \frac{1}{4} \begin{bmatrix} (2 - 3\xi + \xi^3) \\ L(1 - \xi - \xi^2 + \xi^3) \\ (2 + 3\xi - \xi^3) \\ L(-1 - \xi + \xi^2 + \xi^3) \end{bmatrix} \begin{cases} 0\,, -2L, \le x < 0 \\ -\frac{Q}{2L} \cdot \frac{x}{L}\,, 0 \le x \le 2L \end{cases} dx
$$

$$\boldsymbol{F}_{b,1} = \int_0^{2L} \frac{1}{4}\begin{bmatrix} (2-3\xi+\xi^3) \\ L(1-\xi-\xi^2+\xi^3) \\ (2+3\xi-\xi^3) \\ L(-1-\xi+\xi^2+\xi^3) \end{bmatrix}\left(-\frac{Q}{2L}\cdot\frac{x}{L}\right)dx$$

In Local Coordinates

$$q(\xi) = \begin{cases} 0\,, -1, \le \xi < 0 \\ -\dfrac{Q}{L}\cdot\xi, 0 \le \xi \le 1 \end{cases}$$

$$\boldsymbol{F}_{b,1} = \int_0^{1} \frac{1}{4}\begin{bmatrix} (2-3\xi+\xi^3) \\ L(1-\xi-\xi^2+\xi^3) \\ (2+3\xi-\xi^3) \\ L(-1-\xi+\xi^2+\xi^3) \end{bmatrix}\left(-\frac{Q}{2L}\cdot\frac{\xi\cdot 2L}{L}\right)d\xi\cdot 2L$$



$$x = \xi\cdot 2L \Leftrightarrow \xi = x/2L$$
$$dx = d\xi\cdot 2L \Leftrightarrow d\xi = dx/2L$$

$$= -\frac{Q}{2}\int_0^{1} \begin{bmatrix} (2-3\xi+\xi^3) \\ L(1-\xi-\xi^2+\xi^3) \\ (2+3\xi-\xi^3) \\ L(-1-\xi+\xi^2+\xi^3) \end{bmatrix}\cdot\xi\cdot d\xi$$

$$= -\frac{Q}{2}\begin{bmatrix} \xi^2-\xi^3+\dfrac{\xi^5}{5} \\ 2L\left(\dfrac{\xi^2}{2}-\dfrac{\xi^3}{3}-\dfrac{\xi^4}{4}+\dfrac{\xi^5}{5}\right) \\ \xi^2+\xi^3-\dfrac{\xi^5}{5} \\ 2L\left(-\dfrac{\xi^2}{2}-\dfrac{\xi^3}{3}+\dfrac{\xi^4}{4}+\dfrac{\xi^5}{5}\right) \end{bmatrix}_0^1 \quad \Rightarrow \boldsymbol{F}_{b,1} = \begin{bmatrix} -Q/10 \\ -7QL/60 \\ -9Q/10 \\ 23QL/60 \end{bmatrix}$$

6) Compute the body force vector for Element 2 (No load)

$$\boldsymbol{F}_{b,2} = \boldsymbol{0}$$

7) Assemble the body force vector

$$\boldsymbol{F} = \boldsymbol{F}_{b,1} + \boldsymbol{F}_{b,2} + \boldsymbol{F}_s \Leftrightarrow \begin{bmatrix} F_1 \\ F_2 \\ M_3 \\ F_4 \\ F_5 \\ M_6 \\ F_7 \\ F_8 \end{bmatrix} = \begin{bmatrix} 0 \\ -Q/10 \\ -7QL/60 \\ 0 \\ -9Q/10 \\ 23QL/60 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ M_{R3} \\ P \\ 0 \\ 0 \\ 0 \\ R_8 \end{bmatrix} = \begin{bmatrix} R_1 \\ -\dfrac{Q}{10}+R_2 \\ -\dfrac{7QL}{60}+M_{R3} \\ P \\ -9Q/10 \\ 23QL/60 \\ 0 \\ R_8 \end{bmatrix}$$

8) Solve the system and find the displacements:

$$\boldsymbol{KD = F}$$

$$\begin{bmatrix} \frac{EA}{4L} & 0 & 0 & -\frac{EA}{4L} & 0 & 0 & 0 & 0 \\ 0 & \frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & -\frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & 0 \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & 0 \\ -\frac{EA}{4L} & 0 & 0 & \frac{EA}{4L} & 0 & 0 & 0 & 0 \\ 0 & -\frac{3EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & \frac{27EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & -\frac{3EI}{2L^3} \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3EI}{2L^3} & 0 & 0 & \frac{3EI}{2L^3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ D_4 \\ D_5 \\ D_6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 - \frac{Q}{10} \\ M_{R3} - \frac{7QL}{60} \\ P \\ -9Q/10 \\ 23QL/60 \\ 0 \\ R_8 \end{bmatrix}$$

$$\begin{bmatrix} \frac{EA}{4L} & 0 & 0 & -\frac{EA}{4L} & 0 & 0 & 0 & 0 \\ 0 & \frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & -\frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & 0 \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & 0 \\ -\frac{EA}{4L} & 0 & 0 & \frac{EA}{4L} & 0 & 0 & 0 & 0 \\ 0 & -\frac{3EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & \frac{27EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & -\frac{3EI}{2L^3} \\ 0 & \frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3EI}{2L^3} & 0 & 0 & \frac{3EI}{2L^3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ D_4 \\ D_5 \\ D_6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 - \frac{Q}{10} \\ M_{R3} - \frac{7QL}{60} \\ P \\ -9Q/10 \\ 23QL/60 \\ 0 \\ R_8 \end{bmatrix}$$

$$\boldsymbol{K_{red} D_{red} = F_{red}}$$

$$\Rightarrow \begin{bmatrix} \frac{EA}{4L} & 0 & 0 \\ 0 & \frac{27EI}{16L^3} & -\frac{3EI}{8L^2} \\ 0 & -\frac{3EI}{8L^2} & \frac{EI}{L} \end{bmatrix} \begin{bmatrix} D_4 \\ D_5 \\ D_6 \end{bmatrix} = \begin{bmatrix} P \\ -\frac{9Q}{10} \\ \frac{23QL}{60} \end{bmatrix} \Rightarrow \begin{bmatrix} D_4 \\ D_5 \\ D_6 \end{bmatrix} = \begin{bmatrix} \frac{4PL}{EA} \\ -\frac{22QL^3}{45EI} \\ \frac{QL^2}{5EI} \end{bmatrix}$$

9) Compute the Forces

$$
\begin{bmatrix}
\frac{EA}{4L} & 0 & 0 & -\frac{EA}{4L} & 0 & 0 & 0 & 0 \\
0 & \frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & -\frac{3EI}{16L^3} & \frac{3EI}{8L^2} & 0 & 0 \\
0 & \frac{3EI}{8L^2} & \frac{EI}{L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & 0 \\
-\frac{EA}{4L} & 0 & 0 & \frac{EA}{4L} & 0 & 0 & 0 & 0 \\
0 & -\frac{3EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & \frac{27EI}{16L^3} & -\frac{3EI}{8L^2} & 0 & -\frac{3EI}{2L^3} \\
0 & \frac{3EI}{8L^2} & \frac{EI}{2L} & 0 & -\frac{3EI}{8L^2} & \frac{EI}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3EI}{2L^3} & 0 & 0 & \frac{3EI}{2L^3}
\end{bmatrix}
\begin{bmatrix}
0 \\ 0 \\ 0 \\ \frac{4PL}{EA} \\ -\frac{22QL^3}{45EI} \\ \frac{QL^2}{5EI} \\ 0 \\ 0
\end{bmatrix}
=
\begin{bmatrix}
R_1 \\ R_2 - \frac{Q}{10} \\ M_{R3} - \frac{7QL}{60} \\ P \\ -9Q/10 \\ 23QL/60 \\ 0 \\ R_8
\end{bmatrix}
$$

$$
\Rightarrow \boldsymbol{F} = \begin{bmatrix} -P \\ Q/6 \\ 17QL/60 \\ P \\ -9Q/10 \\ 23QL/60 \\ 0 \\ 11Q/15 \end{bmatrix}
$$

Reactions:

$R_1 = -P$
$R_2 = 7Q/60$
$M_{R3} = 2QL/5$
$R_8 = 11Q/15$

Equilibrium Check:

$$\sum F_x = -P + P + 0 = 0$$

$$\sum F_y = \frac{Q}{6} + \frac{-9Q}{10} + \frac{11Q}{15} = 0$$

$$\sum M = \frac{17QL}{60} + \frac{23QL}{60} - (4L)\cdot\frac{9Q}{10} + (4L)\cdot\frac{11Q}{15} = 0$$

10) Evaluate the FEM Displacement Function

$$w(\xi) = \boldsymbol{N}\boldsymbol{d}_e$$

$$w(\xi) = \boxed{[(2-3\xi+\xi^3)/4 \quad L(1-\xi-\xi^2+\xi^3)/4 \quad (2+3\xi-\xi^3)/4 \quad L(-1-\xi+\xi^2+\xi^3)/4]}\boxed{\begin{bmatrix} 0 \\ 0 \\ -\frac{22QL^3}{45EI} \\ \frac{QL^2}{5EI} \end{bmatrix}}$$

$$\Rightarrow w(\xi) = \frac{QL^3}{EI}\left(-\frac{62}{45} - \frac{28}{15}\xi + \frac{2}{5}\xi^2 + \frac{8}{9}\xi^3\right)$$

$$\Rightarrow w(x) = \frac{QL^3}{EI}\left(-\frac{62}{45} - \frac{14}{15}\cdot\frac{x}{L} + \frac{1}{10}\frac{x^2}{L^2} + \frac{1}{9}\frac{x^3}{L^3}\right)$$

11) We can evaluate the Bending Moment function

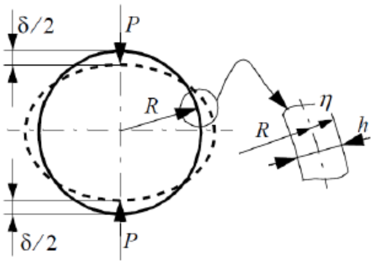
$$M(x) = -EIw''(x)$$

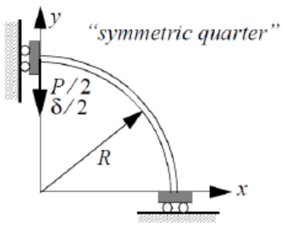
$$\Rightarrow M(x) = -EI\frac{d}{dx^2}\left(\frac{QL^3}{EI}\left(-\frac{62}{45} - \frac{14}{15}\cdot\frac{x}{L} + \frac{1}{10}\frac{x^2}{L^2} + \frac{1}{9}\frac{x^3}{L^3}\right)\right) = QL\left(\frac{1}{5} + \frac{2}{3}\cdot\frac{x}{L}\right)$$

Problem 5.6

Given

- Radius R
- Thickness h
- Elastic Modulus E
- Bending Stiffness EI
- Spring Stifness k
- force P
- If $R \gg h$ and δ depends on bending deformation:
  - $k = \frac{4\pi}{(\pi^2-8)}\frac{EI}{R}$



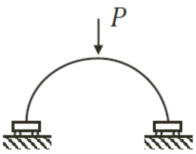


Find

equivalent stiffness using FEM

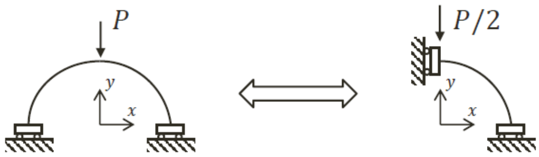
$$k = P/\delta$$

Conditions on Symmetry



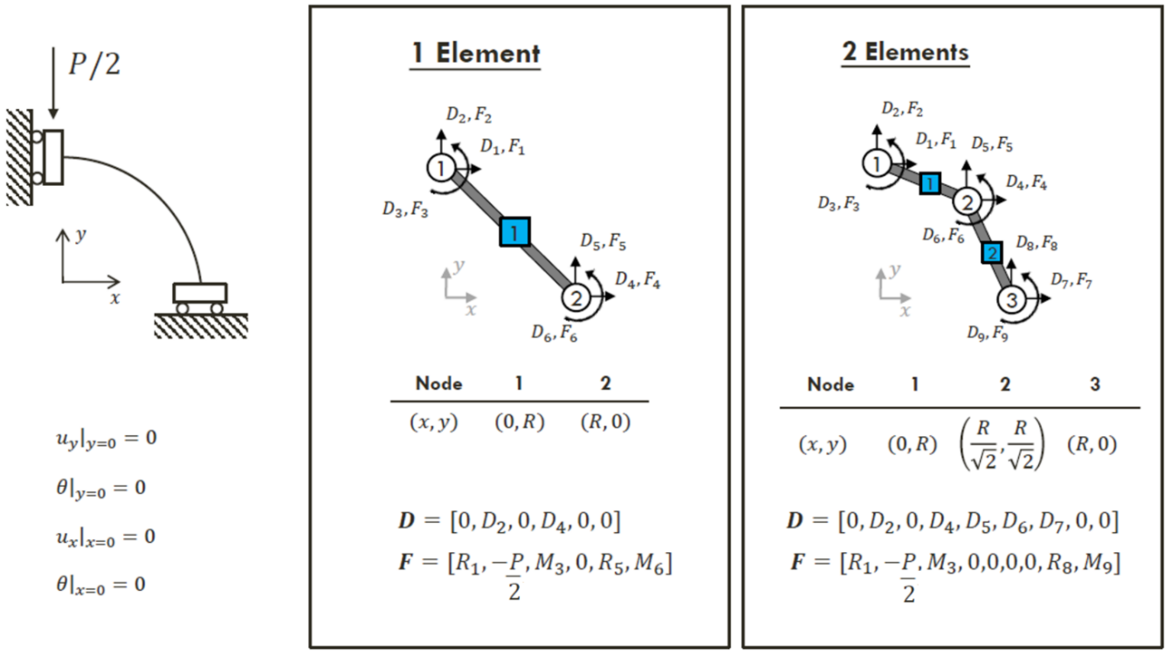
$$u_y\Big|_{y=0} = 0$$

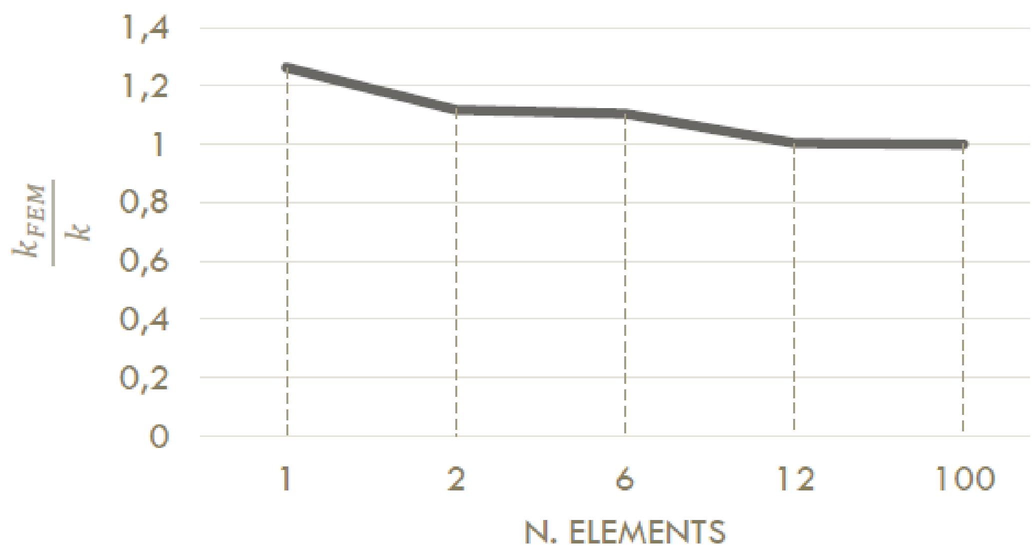
$$\theta\Big|_{y=0} = 0$$



$$u_x\Big|_{x=0} = 0$$

$$\theta\Big|_{x=0} = 0$$


P/2
y
x
u_y|_{y=0} = 0
θ|_{y=0} = 0
u_x|_{x=0} = 0
θ|_{x=0} = 0
1 Element
D_2, F_2
D_1, F_1
D_3, F_3
D_5, F_5
D_4, F_4
D_6, F_6
Node 1 2
(x, y) (0, R) (R, 0)
D = [0, D_2, 0, D_4, 0, 0]
F = [R_1, −P/2, M_3, 0, R_5, M_6]
2 Elements
D_2, F_2
D_1, F_1
D_5, F_5
D_4, F_4
D_3, F_3
D_6, F_6
D_8, F_8
D_7, F_7
D_9, F_9
Node 1 2 3
(x, y) (0, R) (R/√2, R/√2) (R, 0)
D = [0, D_2, 0, D_4, D_5, D_6, D_7, 0, 0]
F = [R_1, −P/2, M_3, 0,0,0,0, R_8, M_9]


1,4
1,2
1
0,8
0,6
0,4
0,2
0
k_FEM / k
1
2
6
12
100
N. ELEMENTS