Tutorial 2: FEM for Engineering Applications (SE1025)

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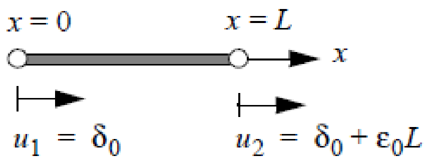
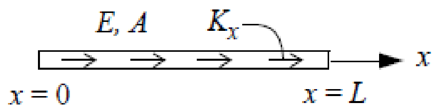
4.1 A uniaxial bar is modelled by a linear truss element. For a certain applied load, the node displacement shown in the figure results. (a) Show that the strain developing in the element is $\varepsilon(x) = \varepsilon_0$ and (b) show that if $\varepsilon_0 = 0$, the element is subjected to a rigid body motion equal to δ_0.



3.3 The figure to the right shows a rod with elastic modulus E and cross sectional area A. The rod is loaded by a body force, K_x [N/m^3]. The displacement, u, in the rod is given by the solution to the differential equation



$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) + AK_x = 0\,.$$

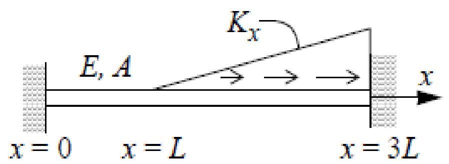
(a) Show that the weak form is $\displaystyle\int_0^L \frac{\mathrm{d}v}{\mathrm{d}x}EA\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int_0^L vK_xA\,\mathrm{d}x + [v(\sigma A)]_0^L$,

where σ denotes the normal stress and v is an arbitrary weight function.

(b) Derive the FEM-equation (use Galerkin's method) to the weak form above for one element, i.e. identify the quantities in the equation

$$\mathbf{k}_e\mathbf{d}_e = \mathbf{f}_e\,.$$

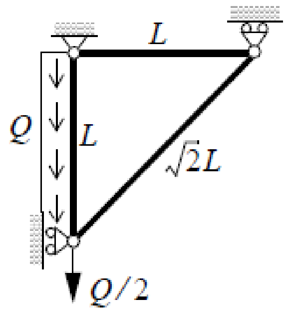
(c) The rod shown to the right is of length $3L$ and loaded by $K_x = Q/(2AL)(x/L-1)$, where Q corresponds to the total axial force acting on the rod. Both ends of the rod are clamped. Divide the rod into two elements of lengths L and 2L respectively and determine the node displacements and the reaction forces. Compare with the exact solution. Redo the analysis with more elements!



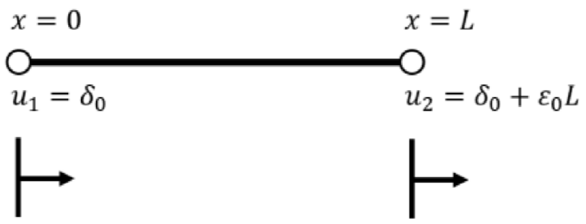
Exact soln.
$$u(x) = \begin{cases} \dfrac{2QLx}{9EAL} & 0 \le x \le L \\ \dfrac{1}{36}\dfrac{QL}{EA}\left[3 - \dfrac{x}{L} + 9\left(\dfrac{x}{L}\right)^2 - 3\left(\dfrac{x}{L}\right)^3\right] & \end{cases}$$

$$N(x=0) = \frac{2Q}{9},\quad N(x=3L) = -\frac{7Q}{9}$$

5.2 A truss structure containing three trusses, all with elastic modulus E and cross sectional area A, is shown to the right. The structure is loaded by one point force $Q/2$ and a body force of total magnitude Q acting on the vertical truss member downwards. Model the structure by use of three linear elements and calculate the displacements and possible reaction forces at the nodes. Note, the displacements at the nodes will in the current case agree with the exact solution. Will the numerical solution deviate from the exact one? If so, how?



Problem 4.1



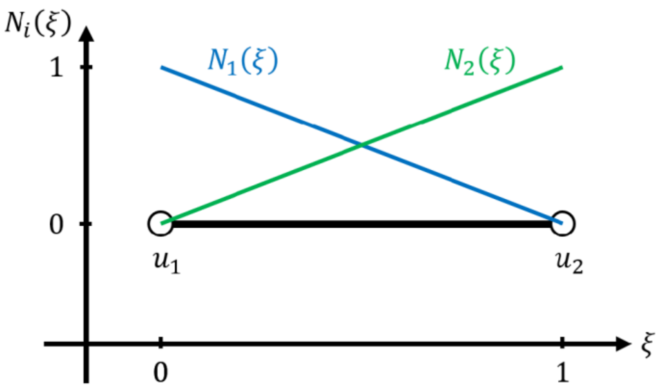
a) Show that the strain is:

$$\varepsilon(x) = \varepsilon_0$$

b) Show that if $\varepsilon_0 = 0$ then

$$u(x) = \delta_0$$

Shape Functions for uniaxial element:



$$N_1(\xi) = 1 - \xi = 1 - \frac{x}{L}$$

$$N_2(\xi) = \xi = \frac{x}{L}$$

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$

$$u(x) = \left(1 - \frac{x}{L}\right)u_1 + \frac{x}{L}u_2$$

a) From the definition of strain from the displacements:

$$\varepsilon_x(x) = \frac{du_x(x)}{dx}$$

$$\varepsilon_x(x) = \frac{d}{dx}\left[\left(1 - \frac{x}{L}\right)u_1 + \frac{x}{L}u_2\right] = -\frac{u_1}{L} + \frac{u_2}{L} = -\frac{\delta_0}{L} + \frac{\delta_0}{L} + \frac{\varepsilon_0 L}{L}$$

$$\boldsymbol{\varepsilon_x(x) = \varepsilon_0}$$

b) From the expression of displacements:

$$u(x) = \left(1 - \frac{x}{L}\right)\boxed{u_1} + \frac{x}{L}\boxed{u_2} = \delta_0$$

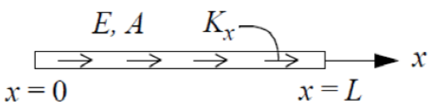
Data

$$u_1 = \delta_0 \qquad u_2 = \delta_0 + \cancel{\varepsilon}_0 L$$

$$\varepsilon_0 = 0$$

$$u_2 = \delta_0$$

Problem 3.3



What we know

- E, A
- Body Force K_x [N/m^3]
- Displacements Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[EA\frac{\mathrm{d}u}{\mathrm{d}x}\right] + AK_x = 0$$

a) Show that the weak form is:

$$\int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx = \int_0^L vK_x A dx + [v(\sigma A)]_0^L$$

b) Derive the FEM- equation (Galerikin's method) for one element:

$$\mathbf{k_e d_e} = \mathbf{f_e}$$

a) General procedure:

From strong to weak formulation

1. Move all expressions to one side of the equation
2. Multiply the equation by the weight function
3. Integrate across the domain
4. Use partial integration (we want to remove the second derivative)
5. Use the boundary conditions of the differential equation

Note.

The strong formulation defines an equation which together with boundary conditions gives the solution for all the points of the domain, while the weak form gives conditions that the integral of the equation must have.

Given the differential equation:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + AK_x = 0$$

(1. Start to manipulate the equation by moving all the terms in one side)

2. Multiply by weight function v

$$\left[\frac{d}{dx}\left(EA\frac{du}{dx}\right) + AK_x\right] \cdot v = 0$$

3. Integrate across the domain

The domain is defined between $0 \leq x \leq L$

$$\int_0^L \left(\left[\frac{d}{dx}\left(EA\frac{du}{dx}\right) + AK_x\right] \cdot v\right) dx = 0$$

$$\int_0^L \left(v \cdot \frac{d}{dx}\left(EA\frac{du}{dx}\right)\right) dx + \int_0^L (v \cdot AK_x) dx = 0$$

4. Integration by parts

$$\underbrace{\int_0^L \left(v \cdot \frac{d}{dx}\left(EA\frac{du}{dx}\right)\right) dx}_{\text{term to be integrated}} + \int_0^L (v \cdot AK_x) dx = 0$$

Note

$$\int_{x_1}^{x_2} g(x)f(x)dx = [g(x)F(x)]_{x_1}^{x_2} - \int_{x_1}^{x_2} g'(x)F(x)dx$$

In our case: $g(x) = v(x)$ $\quad f(x) = \frac{d}{dx}\left(EA\frac{du}{dx}\right)$

$g'(x) = \frac{dv}{dx}$ $\quad F(x) = \int f(x)dx = EA\frac{du}{dx}$

$$\Rightarrow \left[\underbrace{v}_{g(x)} \cdot \underbrace{EA\frac{du}{dx}}_{F(x)}\right]_0^L - \int_0^L \underbrace{\frac{dv}{dx}}_{g'(x)} \cdot \underbrace{EA\frac{du}{dx}}_{F(x)} dx + \int_0^L vAK_x dx = 0$$

$$\Leftrightarrow \int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx = \int_0^L vK_xAdx + \left[vEA\frac{du}{dx}\right]_0^L$$

Using the constitutive relationship: $\sigma = E\varepsilon = E\dfrac{du}{dx} \Leftrightarrow \sigma A = EA\dfrac{du}{dx}$

we finally get :

$$\int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx = \int_0^L vK_x A dx + [v\sigma A]_0^L$$

b) Galerkine's method uses the same shape functions for $\tilde{u}$ and $\tilde{v}$

Terms we need

$$\int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx = \int_0^L vK_x A dx + [v\sigma A]_0^L$$

For a truss element:

- Expression for u(x) and its derivative:

$$u(x) = \sum_{I=1}^{n} N_I u_I = N_1 u_1 + N_2 u_2 = [N_1 \quad N_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \boldsymbol{N d_e}$$

$$\frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 = \underbrace{\begin{bmatrix} \dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} \end{bmatrix}}_{\boldsymbol{B(x)}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\boldsymbol{d_e}} = \frac{d\boldsymbol{N}}{dx} \boldsymbol{d_e} = \boldsymbol{B d_e}$$

- Expression for v(x) and its derivative (we use the same shape functions):

$$v = N_1 \beta_1 + N_2 \beta_2 = [N_1 \quad N_2] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \boldsymbol{N\beta} = \boldsymbol{\beta}^T \boldsymbol{N}^T$$

$$\frac{dv}{dx} = \frac{dN_1}{dx} \beta_1 + \frac{dN_2}{dx} \beta_2 = \underbrace{\begin{bmatrix} \dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} \end{bmatrix}}_{\boldsymbol{B(x)}} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\boldsymbol{\beta}} = \frac{d\boldsymbol{N}}{dx} \boldsymbol{\beta} = \boldsymbol{B\beta} = \boldsymbol{\beta}^T \boldsymbol{B}^T$$

- Now we substitute these expressions in the weak form:

$$\int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx = \int_0^L v K_x A dx + [v\sigma A]_0^L$$

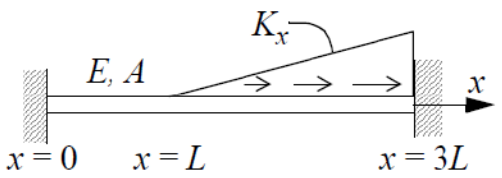
$$\Leftrightarrow \int_{L_e} \boldsymbol{\beta}^T \boldsymbol{B}^T EA\boldsymbol{B}\boldsymbol{d}_e dx = \int_{L_e} \boldsymbol{\beta}^T \boldsymbol{N}^T K_x A dx + [\boldsymbol{\beta}^T \boldsymbol{N}^T \sigma A]_{x1}^{x2}$$

$$\Leftrightarrow \boldsymbol{\beta}^T \int_{L_e} \boldsymbol{B}^T EA\boldsymbol{B} dx \boldsymbol{d}_e = \boldsymbol{\beta}^T \left(\int_{L_e} \boldsymbol{N}^T K_x A dx + [\boldsymbol{N}^T \sigma A]_{x1}^{x2} \right)$$

$$\Leftrightarrow \int_{L_e} \boldsymbol{B}^T EA\boldsymbol{B} dx \boldsymbol{d}_e = \int_{L_e} \boldsymbol{N}^T K_x A dx + [\boldsymbol{N}^T \sigma A]_{x1}^{x2}$$

$$\Downarrow$$

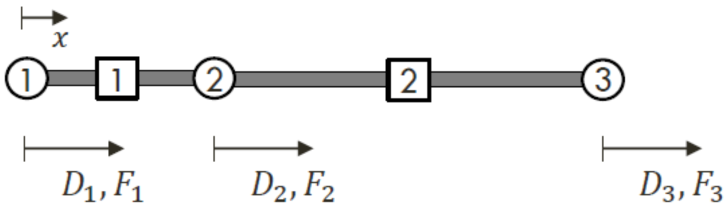
$$\underbrace{\int_{L_e} \boldsymbol{B}^T EA\boldsymbol{B} dx}_{\boldsymbol{k_e}} \underbrace{\boldsymbol{d_e}}_{\boldsymbol{d_e}} = \underbrace{\underbrace{\int_{L_e} \boldsymbol{N}^T K_x A dx}_{\boldsymbol{f_b} - \text{body force}} + \underbrace{[\boldsymbol{N}^T \sigma A]_{x1}^{x2}}_{\boldsymbol{f_s} - \text{point force}}}_{\boldsymbol{f_e}} \Leftrightarrow \boldsymbol{k_e}\boldsymbol{d_e} = \boldsymbol{f_e}$$



c) Given the following expression for the load

$$K_x = \frac{Q}{2AL}\left(\frac{x}{L} - 1\right)$$

Divide the rod into two elements of lengths L and 2L respectively and determine node displacements and reaction forces.



$$\begin{cases} D_1 = 0 \\ D_2 = ? \\ D_3 = 0 \end{cases} \quad \begin{cases} F_1 = ? \\ F_2 = ? \\ F_3 = ? \end{cases}$$

$$\mathbf{KD} = \mathbf{F} = \mathbf{F_b} + \mathbf{F_s}$$

$$\left(\int_0^L \mathbf{B^T} EA\mathbf{B} dx\right)\mathbf{u} = \int_0^L \mathbf{N^T} AK_x dx + [\mathbf{N^T} A\sigma(x)]_0^L$$

1) Determine the Element Stiffness Matrix:

$$\Rightarrow \boldsymbol{k_e} = \int_{L_e} \boldsymbol{B}^T EA\boldsymbol{B} dx \Leftrightarrow \boldsymbol{k_e} = EA \int_{L_e} \boldsymbol{B}^T \boldsymbol{B} dx$$

$$\boldsymbol{B} = \frac{d\mathbf{N}}{dx} \qquad \boldsymbol{B}^T = \frac{d\boldsymbol{N}^T}{dx} \quad \Rightarrow \quad \mathbf{N} = [N_1 \quad N_2] \quad N_1 = 1 - \xi \quad \Rightarrow N_1 = 1 - \frac{x_e}{L_e}$$

$$N_2 = \xi \quad \Rightarrow N_2 = \frac{x_e}{L_e}$$

$$\boldsymbol{B} = \left[-\frac{1}{L_e} \quad \frac{1}{L_e}\right]$$

$$\int_0^{L_e} \boldsymbol{B}^T \boldsymbol{B} dx = \frac{1}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{\Rightarrow \boldsymbol{k_e} = \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

(like the spring with k= EA/L)

We have 2 elements and 3 nodes. Each node has only one degree of freedom

We create the stiffness matrix for each element

Element	Node	Element Stiffness Matrix
1	1 and 2	$k_{e,1} = \frac{EA}{L} \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$
2	2 and 3	$k_{e,2} = \frac{EA}{L} \begin{array}{cc} 2 & 3 \\ \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$

2) Assemble the Stiffness Matrix:

$$K = \frac{EA}{L} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \frac{EA}{L} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$\Leftrightarrow K = \frac{EA}{2L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

3) Determine the body force vectors:

For each element

$$F_b = \int_0^L \mathbf{N}^{\mathbf{T}} A K_x dx \longrightarrow \int_{L_e} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} A K_x \, dx$$

For Element 1:

$$K_x = 0 \rightarrow F_{b,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{array}{l} \textbf{Node 1} \\ \textbf{Node 2} \end{array}$$

For Element 2:

$$K_x = \frac{Q}{2AL}\left(\frac{x}{L} - 1\right)$$

$$\boxed{\boldsymbol{N}^T = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \quad \text{in local coordinates}}$$

$$\boldsymbol{F}_{\boldsymbol{b},\boldsymbol{2}} = \int_{L_e} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \frac{Q}{2AL}\left(\frac{x}{L}-1\right) A dx = \int_{L}^{3L} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \frac{Q}{2L}\left(\frac{x}{L}-1\right) dx$$

We need to find the relationship between local and global coordinates:

$$\begin{cases} x = k\xi + m \\ x = L, \text{for } \xi = 0 \\ x = 3L, \text{for } \xi = 1 \end{cases} \Rightarrow \begin{cases} m = L \\ k = 2L \end{cases} \Rightarrow \begin{cases} x = 2L\xi + L \\ dx = 2L \cdot d\xi \end{cases} \Rightarrow \begin{cases} \xi = \frac{1}{2}\left(\frac{x}{L}-1\right) \\ d\xi = \frac{1}{2L} dx \end{cases}$$

It is easier to integrate in local coordinates:

$$\boldsymbol{F}_{\boldsymbol{b},\boldsymbol{2}} = \int_{L}^{3L} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \frac{Q}{2L}\left(\frac{x}{L}-1\right) dx = \int_{0}^{1} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \frac{Q}{2L}\left(\frac{2L\xi + L}{L}-1\right) 2L \cdot d\xi$$

$$\Leftrightarrow \boldsymbol{F}_{\boldsymbol{b},\boldsymbol{2}} = \frac{Q}{3}\begin{array}{c|l} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{matrix} \text{Node 2} \\ \text{Node 3} \end{matrix} \end{array}$$

4) Assembly body force vectors:

$$\boldsymbol{F}_{\boldsymbol{b},\boldsymbol{1}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \end{matrix} \qquad \boldsymbol{F}_{\boldsymbol{b},\boldsymbol{2}} = \frac{Q}{3}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{matrix} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \end{matrix}$$

5) Sum the nodal force vectors in the corresponding nodes (reactions)

$$\boldsymbol{F} = \boldsymbol{F}_{\boldsymbol{b},\boldsymbol{1}} + \boldsymbol{F}_{\boldsymbol{b},\boldsymbol{2}} + \boldsymbol{F}_{\boldsymbol{s}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{Q}{3}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} R_1 \\ 0 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ Q/3 \\ 2Q/3 + R_3 \end{bmatrix}$$

6) Solve the system and find the displacements

$$\boldsymbol{KD} = \boldsymbol{F}$$

$$\frac{EA}{2L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ D_2 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ Q/3 \\ 2Q/3 + R_3 \end{bmatrix}$$

only D_2 is unknown

$$\frac{EA}{2L}\cdot 3\cdot D_2=\frac{Q}{3}\Leftrightarrow D_2=\frac{2QL}{9EA}\Leftrightarrow\begin{bmatrix}D_1\\D_2\\D_3\end{bmatrix}=\begin{bmatrix}0\\2QL/9EA\\0\end{bmatrix}$$

7) Find the forces

$$\boldsymbol{KD}=\boldsymbol{F}$$

$$\Leftrightarrow\frac{EA}{2L}\begin{bmatrix}2&-2&0\\-2&3&-1\\0&-1&1\end{bmatrix}\begin{bmatrix}0\\2QL/9EA\\0\end{bmatrix}=\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}=\begin{bmatrix}R_1\\Q/3\\2Q/3+R_3\end{bmatrix}$$

$$\Leftrightarrow\begin{bmatrix}2&-2&0\\-2&3&-1\\0&-1&1\end{bmatrix}\frac{Q}{9}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}R_1\\Q/3\\2Q/3+R_3\end{bmatrix}$$

$$\Leftrightarrow\frac{Q}{9}\begin{bmatrix}2\cdot 0-2\cdot 1+0\cdot 0\\-2\cdot 0+3\cdot 1-1\cdot 0\\0\cdot 0-1\cdot 1+1\cdot 0\end{bmatrix}=\begin{bmatrix}R_1\\Q/3\\2Q/3+R_3\end{bmatrix}$$

$$\Leftrightarrow\frac{Q}{9}\begin{bmatrix}-2\\3\\-1\end{bmatrix}=\begin{bmatrix}R_1\\Q/3\\2Q/3+R_3\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}=\begin{bmatrix}R_1\\Q/3\\2Q/3+R_3\end{bmatrix}=\begin{bmatrix}-2Q/9\\3Q/9\\-Q/9\end{bmatrix}$$

$$\boxed{\Rightarrow\begin{cases}R_1=-2Q/9\\R_3=-7Q/9\end{cases}}$$

Note that

$$\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}=\begin{bmatrix}-2Q/9\\Q/3\\-Q/9\end{bmatrix}\Leftrightarrow\begin{bmatrix}R_1\\R_2\\R_3\end{bmatrix}=\begin{bmatrix}-2Q/9\\0\\-7Q/9\end{bmatrix}$$

Check equilibrium:

$$\sum F=0\Leftrightarrow-\frac{2Q}{9}+\frac{Q\cdot 3}{3\cdot 3}-\frac{Q}{9}=0$$ ok!

8) Compare with the real solution:

Real Solution:

$$u(x) = \begin{cases} \dfrac{2QL}{9EA} \cdot \dfrac{x}{L} & 0 \le x \le L \\ \dfrac{QL}{36EA}\left[3 - \dfrac{x}{L} + 9\left(\dfrac{x}{L}\right)^2 - 3\left(\dfrac{x}{L}\right)^3\right] & L \le x \le 3L \end{cases}$$

FEM Solution

$$u(x) = \begin{cases} N_1^{(1)}(x)u_1 + N_2^{(1)}(x)u_2 & 0 \le x \le L \\ N_1^{(2)}(x)u_2 + N_2^{(2)}(x)u_3 & L \le x \le 3L \end{cases} \qquad \begin{bmatrix} N_1^{(1)} \\ N_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \quad \xi = \frac{x}{L} \qquad \begin{bmatrix} N_1^{(2)} \\ N_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \quad \xi = \frac{1}{2}\left(\frac{x}{L} - 1\right)$$

$u_1 = 0$

$$u(x) = \begin{cases} \cancel{N_1^{(1)}(x)u_1} + N_2^{(1)}(x)u_2 & 0 \le x \le L \\ N_1^{(2)}(x)u_2 + \cancel{N_2^{(2)}(x)u_3} & L \le x \le 3L \end{cases}$$

$u_3 = 0$

$u_1 = 0$

$$u(x) = \begin{cases} \cancel{N_1^{(1)}(x)u_1} + N_2^{(1)}(x)u_2 & 0 \le x \le L \\ N_1^{(2)}(x)u_2 + \cancel{N_2^{(2)}(x)u_3} & L \le x \le 3L \end{cases}$$

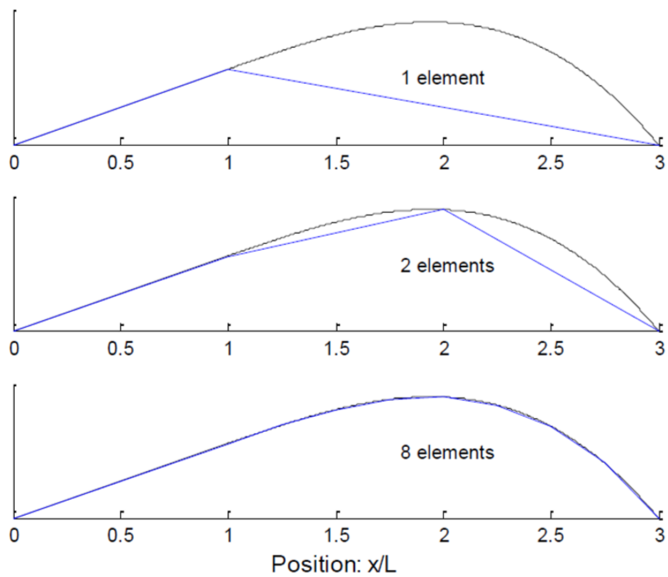
$u_3 = 0$

$$u(x) = \begin{cases} \dfrac{x}{L} \quad 2QL/9EA & 0 \le x \le L \\ \left[1 - \dfrac{1}{2}\left(\dfrac{x}{L} - 1\right)\right] 2QL/9EA & L \le x \le 3L \end{cases}$$

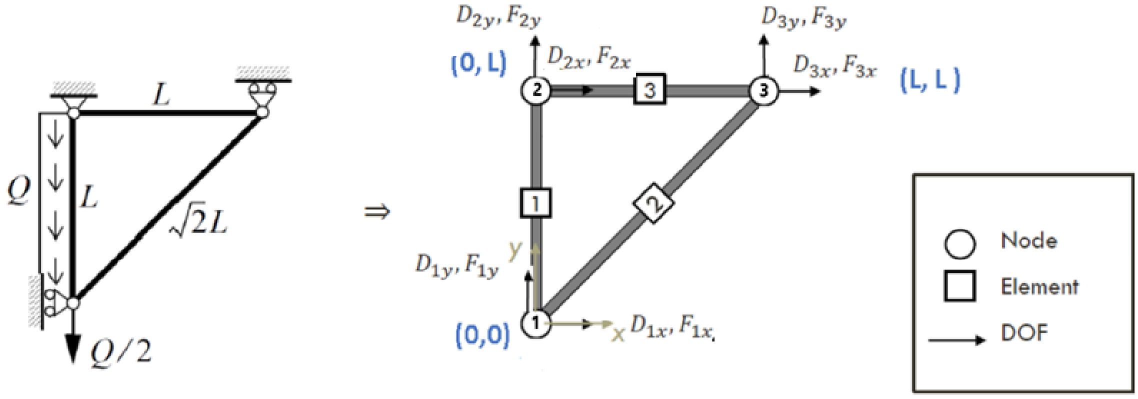
can capture linear behavior

cannot caputure more

At nodes, both forces and displacements are captured!


1 element
2 elements
8 elements
0
0.5
1
1.5
2
2.5
3
Position: x/L

Problem 5.2



Data:

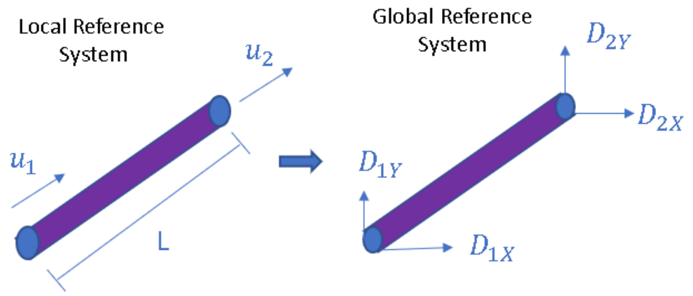
Forces

$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix} = \begin{bmatrix} R_{1x} \\ -Q/2 \\ R_{2x} \\ R_{2y} \\ 0 \\ R_{3y} \end{bmatrix}$$

Displacements

$$\begin{bmatrix} D_{1x} \\ D_{1y} \\ D_{2x} \\ D_{2y} \\ D_{3x} \\ D_{3y} \end{bmatrix} = \begin{bmatrix} 0 \\ D_{1y} \\ 0 \\ 0 \\ D_{3x} \\ 0 \end{bmatrix}$$

1) Compute the elements' stiffness matrix



$$\boldsymbol{K}_E = \frac{EA}{L}\begin{bmatrix} \boldsymbol{a} & -\boldsymbol{a} \\ -\boldsymbol{a} & \boldsymbol{a} \end{bmatrix}$$

$$\boldsymbol{a} = \begin{bmatrix} l_{12}^2 & l_{12}m_{12} \\ l_{12}m_{12} & m_{12}^2 \end{bmatrix}$$

Element 1

Element	Nodes	DOF	$l_{12} = \frac{x_2 - x_1}{l}$	$m_{12} = \frac{y_2 - y_1}{l}$	$\boldsymbol{a}$
1	1-2	$\boldsymbol{D_{1X}}\ \boldsymbol{D_{1Y}}$ $\boldsymbol{D_{2X}}\ \boldsymbol{D_{2Y}}$	0	1	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$
\Rightarrow \mathbf{k}_{e1} = \frac{EA}{L}
\begin{array}{c}
\begin{array}{cccc} D_{1X} & D_{1Y} & D_{2X} & D_{2Y} \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{array}
\begin{array}{c} D_{1X} \\ D_{1Y} \\ D_{2X} \\ D_{2Y} \end{array}
$$

Element 2

Element	Nodes	DOF	$l_{12} = \frac{x_2 - x_1}{l}$	$m_{12} = \frac{y_2 - y_1}{l}$	$\boldsymbol{a}$
2	1-3	$D_{1X}\ D_{1Y}$ $D_{3X}\ D_{3Y}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$
\Rightarrow \mathbf{k}_{e2} = \frac{EA}{\sqrt{2}L} \cdot \frac{1}{2}
\begin{array}{c}
\begin{array}{cccc} D_{1X} & D_{1Y} & D_{3X} & D_{3Y} \end{array} \\
\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}
\end{array}
\begin{array}{c} D_{1X} \\ D_{1Y} \\ D_{3X} \\ D_{3Y} \end{array}
$$

Element 3

Element	Nodes	DOF	$l_{12} = \frac{x_2 - x_1}{l}$	$m_{12} = \frac{y_2 - y_1}{l}$	$\boldsymbol{a}$
3	2-3	$D_{2X}\ D_{2Y}$ $D_{3X}\ D_{3Y}$	1	0	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$
\Rightarrow \mathbf{k}_{e3} = \frac{EA}{L}
\begin{array}{c}
\begin{array}{cccc} D_{2X} & D_{2Y} & D_{3X} & D_{3Y} \end{array} \\
\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} D_{2X} \\ D_{2Y} \\ D_{3X} \\ D_{3Y} \end{array}
$$

2) Assemble the Global Stiffness Matrix

Element	Nodes	Element Stiffness Matrix
1	1 and 2	$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$
2	1 and 3	$\frac{EA}{L}\begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}$
3	2 and 3	$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$
\mathbf{K} = \frac{EA}{2\sqrt{2}L}
\begin{array}{c}
\begin{array}{cccccc} D_{1X} & D_{1Y} & D_{2X} & D_{2Y} & D_{3X} & D_{3Y} \end{array} \\
\begin{bmatrix}
1+0 & 1+0 & 0 & 0 & -1 & -1 \\
1+0 & 1+2\sqrt{2} & 0 & -2\sqrt{2} & -1 & -1 \\
0 & 0 & 0+2\sqrt{2} & 0+0 & -2\sqrt{2} & 0 \\
0 & -2\sqrt{2} & 0+0 & 2\sqrt{2}+0 & 0 & 0 \\
-1 & -1 & -2\sqrt{2} & 0 & 1+2\sqrt{2} & 1+0 \\
-1 & -1 & 0 & 0 & 1+0 & 1+0
\end{bmatrix}
\end{array}
\begin{array}{c} D_{1X} \\ D_{1Y} \\ D_{2X} \\ D_{2Y} \\ D_{3X} \\ D_{3Y} \end{array}
$$

$$\mathbf{K}=\frac{EA}{2\sqrt{2}L}\begin{bmatrix}1 & 1 & 0 & 0 & -1 & -1\\ 1 & 1+2\sqrt{2} & 0 & -2\sqrt{2} & -1 & -1\\ 0 & 0 & 2\sqrt{2} & 0 & -2\sqrt{2} & 0\\ 0 & -2\sqrt{2} & 0 & 2\sqrt{2} & 0 & 0\\ -1 & -1 & -2\sqrt{2} & 0 & 1+2\sqrt{2} & 1\\ -1 & -1 & 0 & 0 & 1 & 1\end{bmatrix}$$

3) Compute the Load Vector

$$\underbrace{\int_0^L \mathbf{N}^{\mathbf{T}} A K_x dx}_{\text{Volume Force}} + \underbrace{[\mathbf{N}^{\mathbf{T}} A\sigma(x)]_0^L}_{\text{Nodal Force}}$$

$$\boldsymbol{F_b} \qquad\qquad \boldsymbol{F_s}$$

Only Element 1 is loaded:

$$\boldsymbol{F_{b,1}} = \int_0^1 \mathbf{N}^{\mathbf{T}} K_x A l_e d\xi \quad \text{with} \quad K_x = -\frac{Q}{AL} \quad l_e = L$$

Integrate in local (element) reference system:

$$\boldsymbol{F_{b,1}} = -\frac{Q}{LA} LA \int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} d\xi \Rightarrow f_e^{(1)} = -\frac{Q}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

It is possible to integrate in global reference system:

$$N = \begin{bmatrix} 1-\frac{x}{L_e} & \frac{x}{L_e} \end{bmatrix} = [1-\xi \quad \xi]$$

$$\boldsymbol{F_{b,1}} = \int_{V_e} N^T K_x dV = \int_0^L \underbrace{\begin{bmatrix} x/L \\ 1-x/L \end{bmatrix}}_{N^T} \underbrace{\frac{-Q}{AL}}_{K} \underbrace{Adx}_{dV} = \frac{-Q}{L^2}\int_0^L \begin{bmatrix} x \\ L-x \end{bmatrix} dx = \frac{-Q}{2L^2}\begin{bmatrix} x^2 \\ 2Lx - x^2 \end{bmatrix}_0^L = \frac{-Q}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

Assembling the load vectors:

$$\Rightarrow \boldsymbol{F_b} = \frac{-Q}{2}\begin{bmatrix}0\\1\\0\\1\\0\\0\end{bmatrix}\begin{matrix}F_{1x}\\F_{1y}\\F_{2x}\\F_{2y}\\F_{3x}\\F_{3y}\end{matrix} \qquad \boldsymbol{F_s} = \begin{bmatrix}R_{1x}\\-Q/2\\R_{2x}\\R_{2y}\\0\\R_{3y}\end{bmatrix}\begin{matrix}F_{1x}\\F_{1y}\\F_{2x}\\F_{2y}\\F_{3x}\\F_{3y}\end{matrix} \qquad \boldsymbol{F} = \boldsymbol{F_b} + \boldsymbol{F_s} = \frac{-Q}{2}\begin{bmatrix}0\\1\\0\\1\\0\\0\end{bmatrix} + \begin{bmatrix}R_{1x}\\-Q/2\\R_{2x}\\R_{2y}\\0\\R_{3y}\end{bmatrix} = \begin{bmatrix}R_{1x}\\-\frac{Q}{2}-\frac{Q}{2}\\R_{2x}\\R_{2y}-\frac{Q}{2}\\0\\R_{3y}\end{bmatrix}$$

4) Solve the system to find the displacements

$$\frac{EA}{2\sqrt{2}L}\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{-1} \\ \cancel{1} & 1+2\sqrt{2} & \cancel{0} & \cancel{-2\sqrt{2}} & -1 & \cancel{-1} \\ \cancel{0} & \cancel{0} & \cancel{2\sqrt{2}} & \cancel{0} & \cancel{-2\sqrt{2}} & \cancel{0} \\ \cancel{0} & \cancel{-2\sqrt{2}} & \cancel{0} & \cancel{2\sqrt{2}} & \cancel{0} & \cancel{0} \\ \cancel{-1} & -1 & \cancel{-2\sqrt{2}} & \cancel{0} & 1+2\sqrt{2} & \cancel{1} \\ \cancel{-1} & \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} \end{bmatrix}\begin{bmatrix} 0 \\ D_{1y} \\ 0 \\ 0 \\ D_{3x} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{1x} \\ -\frac{Q}{2}-\frac{Q}{2} \\ R_{2x} \\ R_{2y}-\frac{Q}{2} \\ 0 \\ R_{3y} \end{bmatrix}$$

$$\mathbf{K}_{\text{red}} = \frac{EA}{2\sqrt{2}L}\begin{bmatrix} 1+2\sqrt{2} & -1 \\ -1 & 1+2\sqrt{2} \end{bmatrix}$$

$$\mathbf{K}_{\text{red}}^{-1} = \frac{1}{EA}\cdot\frac{2\sqrt{2}L}{\left(1+2\sqrt{2}\right)^2-1}\begin{bmatrix} 1+2\sqrt{2} & 1 \\ 1 & 1+2\sqrt{2} \end{bmatrix} \qquad \mathbf{K}_{\text{red}}^{-1} = \frac{1}{EA}\cdot\frac{2\sqrt{2}L}{8+4\sqrt{2}}\begin{bmatrix} 1+2\sqrt{2} & 1 \\ 1 & 1+2\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} D_{1y} \\ D_{3x} \end{bmatrix} = \frac{1}{EA}\cdot\frac{2\sqrt{2}L}{8+4\sqrt{2}}\begin{bmatrix} 1+2\sqrt{2} & 1 \\ 1 & 1+2\sqrt{2} \end{bmatrix}\begin{bmatrix} -Q \\ 0 \end{bmatrix}$$

$$\boldsymbol{D_{1y} = -\frac{2\sqrt{2}L\left(1+2\sqrt{2}\right)Q}{EA\left(8+4\sqrt{2}\right)}} \quad \boldsymbol{D_{3x} = -\frac{2\sqrt{2}LQ}{EA\left(8+4\sqrt{2}\right)}}$$

6) Compute the Reactions:

$$\frac{EA}{2\sqrt{2}L}\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1+2\sqrt{2} & 0 & -2\sqrt{2} & -1 & -1 \\ 0 & 0 & 2\sqrt{2} & 0 & -2\sqrt{2} & 0 \\ 0 & -2\sqrt{2} & 0 & 2\sqrt{2} & 0 & 0 \\ -1 & -1 & -2\sqrt{2} & 0 & 1+2\sqrt{2} & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}\frac{-2\sqrt{2}LQ}{EA\left(8+4\sqrt{2}\right)}\begin{bmatrix} 0 \\ 1+2\sqrt{2} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{1x} \\ -\frac{Q}{2}-\frac{Q}{2} \\ R_{2x} \\ R_{2y}-\frac{Q}{2} \\ 0 \\ R_{3y} \end{bmatrix}$$

$$R_{1x} = \frac{EA}{2\sqrt{2}L}\cdot\frac{-2\sqrt{2}LQ}{EA\left(8+4\sqrt{2}\right)}\left[1+2\sqrt{2}-1\right] = \frac{-2\sqrt{2}Q}{\left(8+4\sqrt{2}\right)}$$

$$R_{2x} = \frac{EA}{2\sqrt{2}L}\cdot\frac{-2\sqrt{2}LQ}{EA\left(8+4\sqrt{2}\right)}\left[0+\left(-2\sqrt{2}\right)\right] = \frac{2\sqrt{2}Q}{\left(8+4\sqrt{2}\right)}$$

$$R_{2y} = \frac{EA}{2\sqrt{2}L}\cdot\frac{-2\sqrt{2}LQ}{EA\left(8+4\sqrt{2}\right)}\left[\left(-2\sqrt{2}\right)\left(1+2\sqrt{2}\right)+0\right]+\frac{Q}{2} = \frac{2\sqrt{2}\left(1+2\sqrt{2}\right)Q}{\left(8+4\sqrt{2}\right)}+\frac{Q}{2}$$

$$R_{3y} = \frac{EA}{2\sqrt{2}L}\cdot\frac{-2\sqrt{2}LQ}{EA\left(8+4\sqrt{2}\right)}\left[-\left(1+2\sqrt{2}\right)+1\right] = \frac{2\sqrt{2}Q}{\left(8+4\sqrt{2}\right)}$$

7) Check:

$$\sum F_x = R_{1x} + R_{2x} = 0, \qquad OK!$$

$$\sum F_y = R_{2y} + R_{3y} - Q - \frac{Q}{2} = 0, \qquad OK!$$