FEM for engineering applications (6 hp-credits)

— continuation course in Solid Mechanics —

Goal: to learn the fundamentals of the *Finite Element Method* (FEM) and how to work with FEM as an engineering tool to solve problems of technical importance

Scheduled teaching

18 Lectures (theory, examples & case studies)

Erik Olsson (coordinator & lecturer)

8 Tutorials (examples & case studies)

Chiara Ceccato and Hossein Shariati

2 Compulsory computer workshops

— solving problems by use of the FEM software ANSYS (a student version can be down loaded)

— Held in the Solid Mechanics track room

— Xarried out in groups of 2 or 3 students

3 Homework assignments

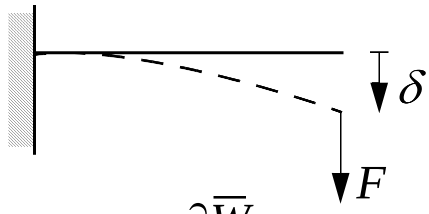
— to carried out in groups of 2 or 3 students

— give bonus points at the written exam

Outline of the course

1. Energy principles and methods
(~ 3 Lectures., 1 Tutorial & 0.5 Home work assign.)

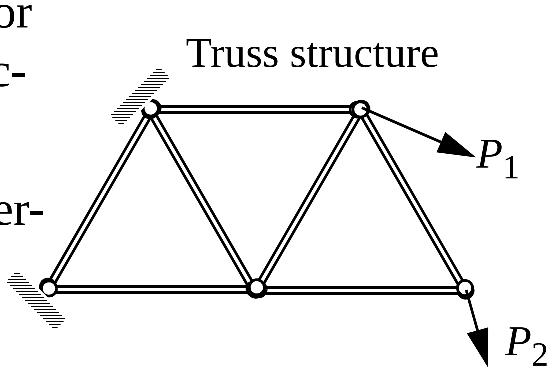
- Fundamental concepts
- Analysis of statically in-determined problems
- Formulation suited for **computational methods**

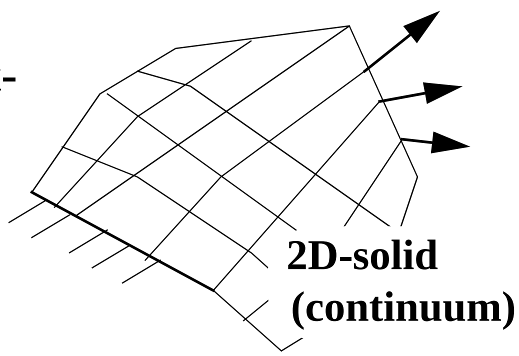


$$\delta = \frac{\partial \overline{W}}{\partial F}$$

2. Finite Element Method
(~ 14 Lect., 6 Tut., 2 Workshops & 1.5 HW)

- Formulation of FEM-equations for structures, solids and heat conduction problems
- Approximate displacement/temperature interpolation for trusses, beams, 2D- and 3D solids
- Matrix formulation—suitable for computational analysis by computers
- FEM-analysis with commercial software used in industry (Workshops)





Literature

- Handouts on *Energy principles and methods*
- *The finite element method—A practical course* (2003) by G.R. Liu & S.S. Quek (available as an E-book at the library KTHB, can also be bought for about 500 SEK)
- *FEM for engineering applications—Exercises with solutions* (Aug. 2008) by Jonas Faleskog

=> Course package containing:

* *FEM for engineering applications—Exercises with solutions*

sold at the student office

Teknikringen 8D prize 100 SEK.

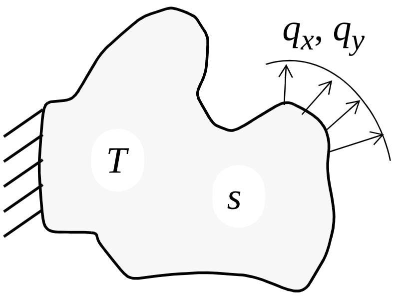
Home page: https://kth.instructure.com/courses/6888

Homework assignments
Instructions for computer workshops
Slides from lectures (pdf-file)
Old exams
Matlab programs: Spring/truss structures
1D Beam problems
Frameworks of beam elements

The Finite Element Method (FEM)

- Many physical phenomenon in engineering and science can be described by ***partial differential equations (PDE)***. These are in general impossible to solve with classical analytical methods.

Steady state heat transfer in 2D (scalar field problem: T)



PDE: $\nabla^T(\mathbf{D}\nabla T) + s = 0$

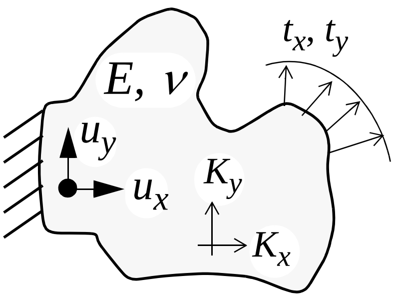
Primary variable
Temperature

$$\nabla = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}$$

Linear elasticity in 2D (vector field problem: u_x & u_y)

PDE: $\nabla_S^T(\mathbf{D}\nabla_S \mathbf{u}) + \mathbf{b} = \mathbf{0}$

Primary variables
displacements



$$\nabla_S = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \qquad \mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} K_x \\ K_y \end{bmatrix}$$

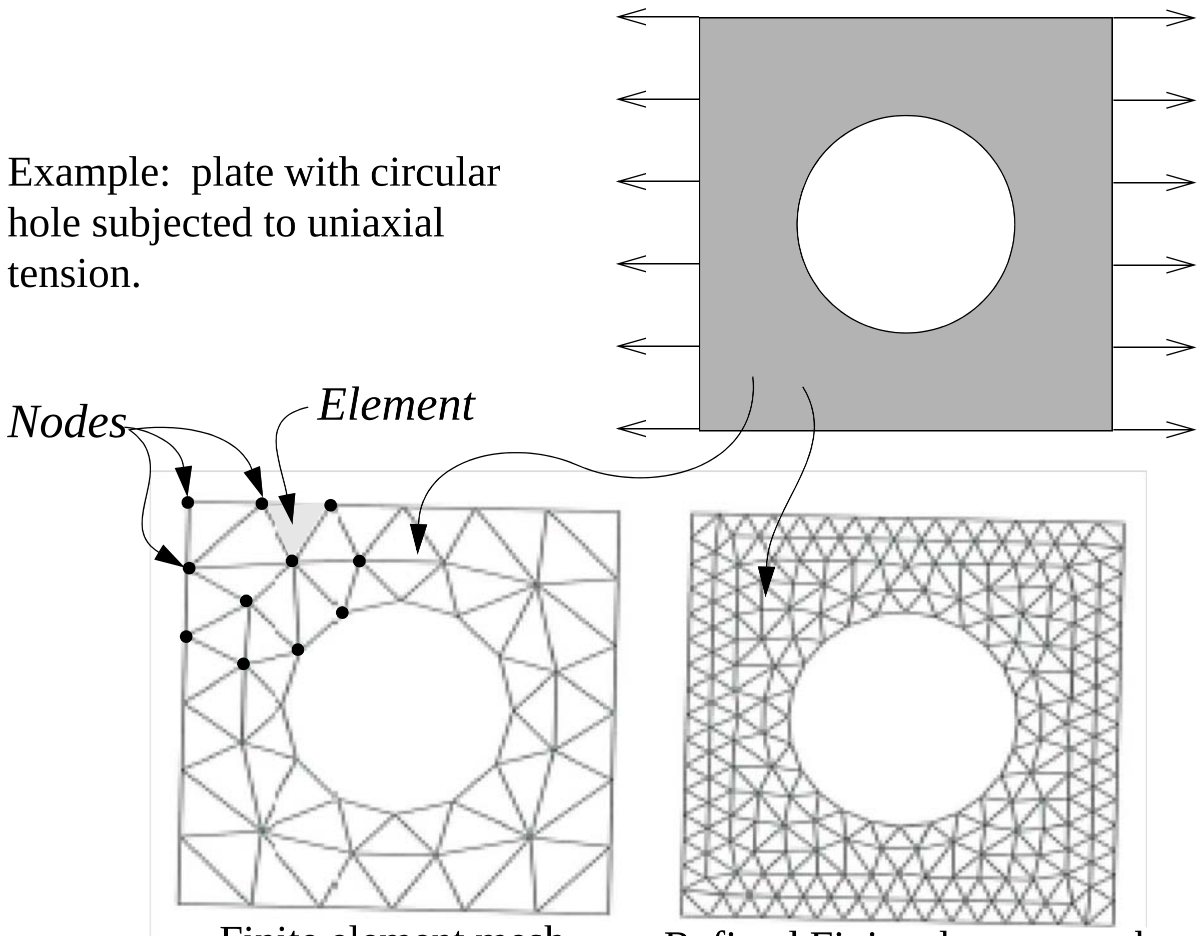
Other examples: Diffusion, Fluid flow, Electromagnetics, etc.

- ***FEM is a numerical approach to approximately solve a PDE***, resulting in a system of linear (or nonlinear) equations in the discrete values of the primary variable/variables which is solved by a computer.

FEM—the basic idea!

- To divide (discretize) the body into ***finite elements***, connected by ***nodes***, and to obtain ***approximate solutions in each element*** (often based on low degree polynomials).
- The "discretized body" is denoted the ***finite element mesh*** and the process of making it is called ***mesh generation***.

Example: plate with circular hole subjected to uniaxial tension.



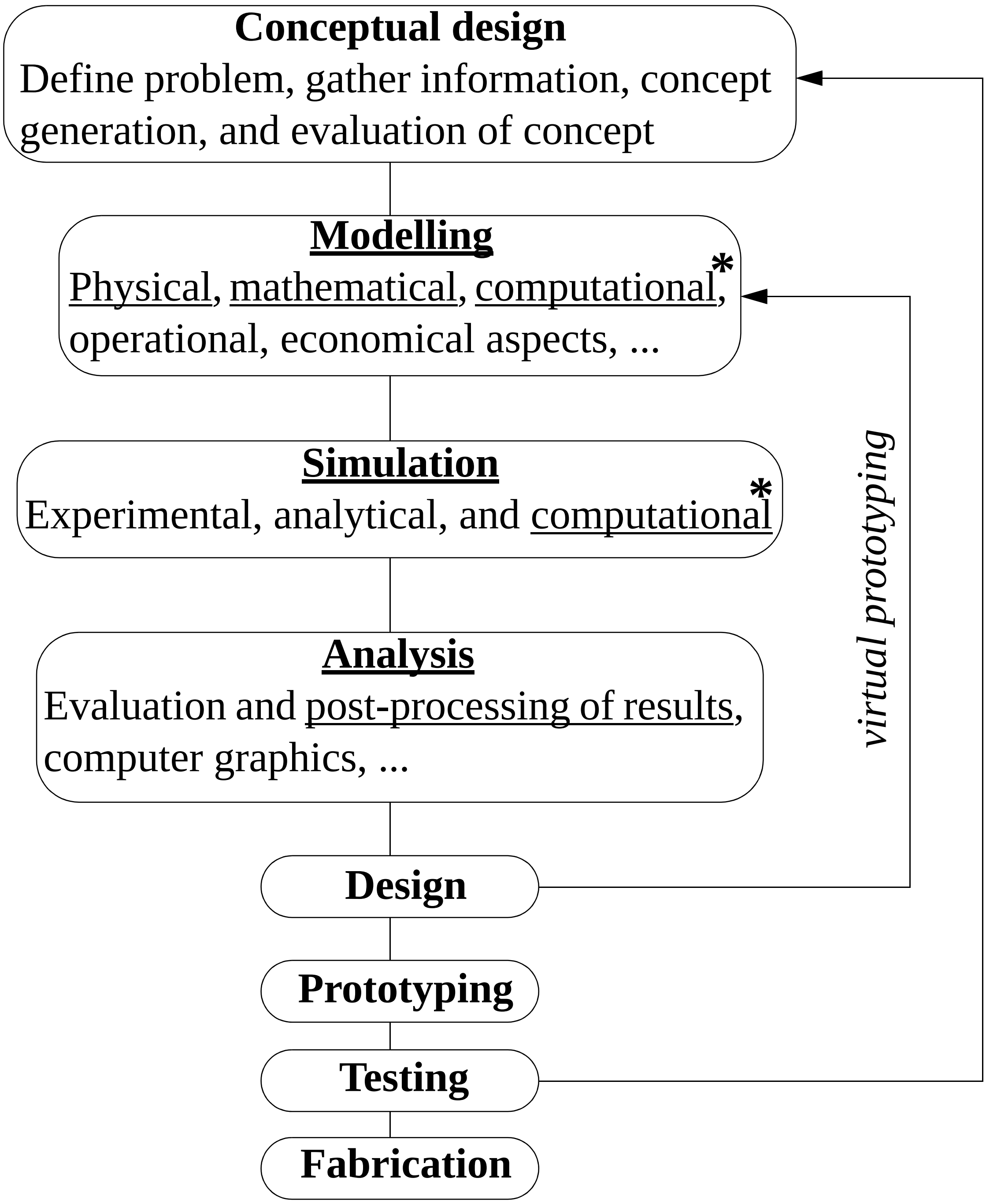
Finite element mesh   Refined Finite element mesh

- The approximate solutions in each element is expressed by use of the ***nodal values of the primary variable/variables***, which comes out as the solution when solving the system of equations. The ***accuracy depends on the size of the elements and number of nodes used.***
- To arrive at the equation system (FEM-Eq.), the PDE (***strong form***) is reformulated into a ***variational form (weak form)***.
- In linear elasticity, the ***Principle of virtual work*** and the ***Theorem of Stationary Energy*** directly leads to the weak form!

FEM in practise

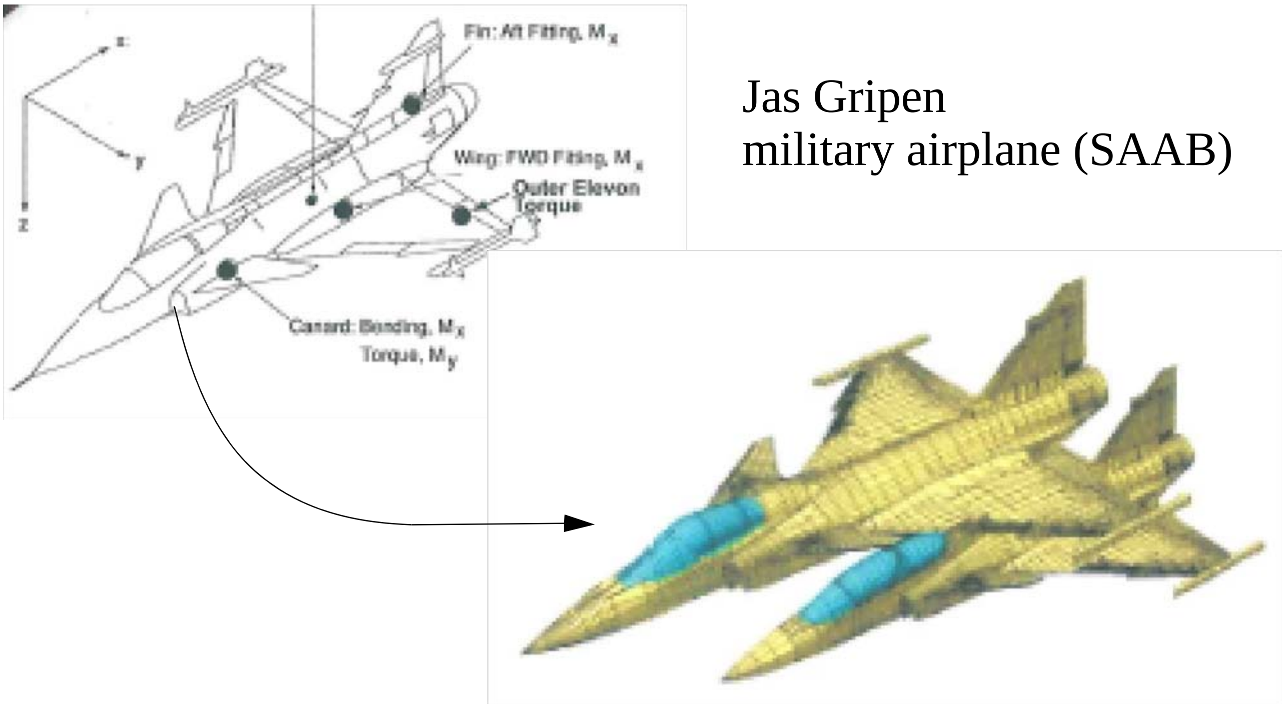
- Used on a regular basis in industry ***to predict the behaviour*** of structural, mechanical, thermal, electrical and chemical systems ***for both design and performance analysis***.

- FEM in the design process for engineering systems:



* Chose a FEM-program, where the FEM-Eq. of the physical phenomenon to be analyzed is implemented. Commercial programs, examples: ANSYS, ABAQUS, NASTRAN, ...

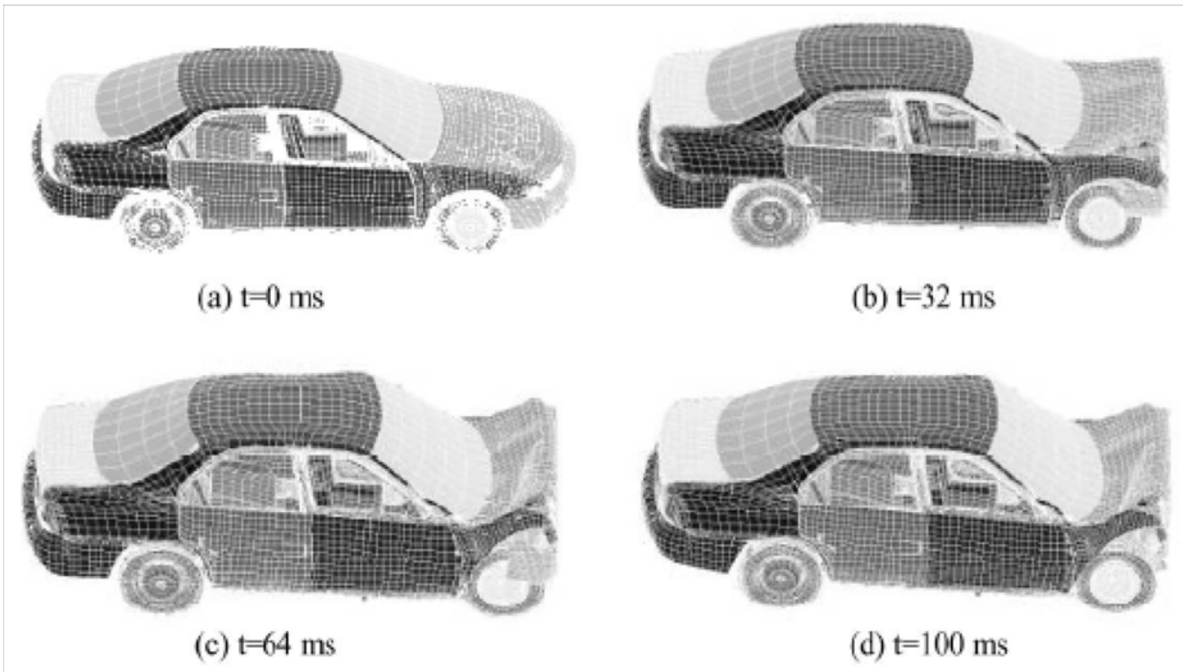
Examples of large structures analysed by FEM



Jas Gripen
military airplane (SAAB)

From: H. Ansell (1998), *Mekanisten* **1998:3**

Crash simulations of an automobile



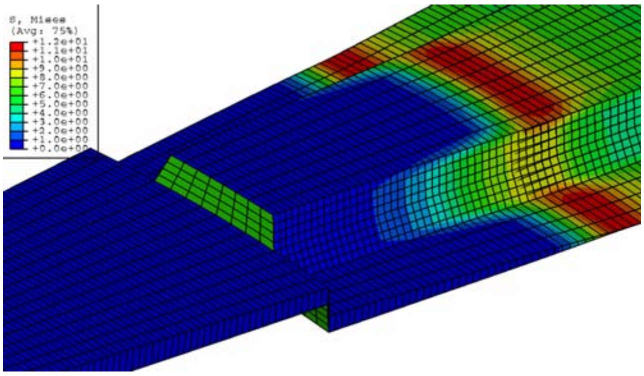
From: Z.Q. Cheng (2001), *Finite Elem. Anal. Design* **37.**

***more examples* — in science ...**

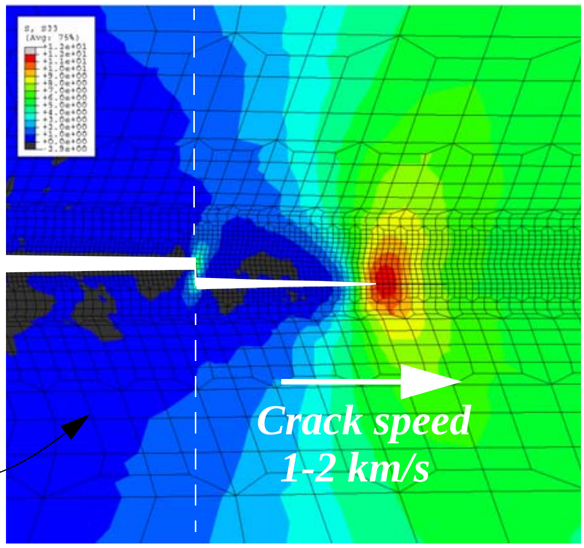
Micromechanical 3D FEM analysis on the micron scale

For development of fracture criteria in structural steels

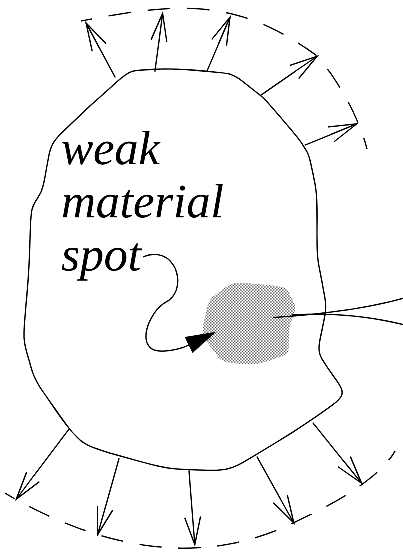
(I. Barsoum, J. Faleskog and M. Stec, KTH Solid Mechanics, 2007)



Deformed mesh of the cleavage planes



Deformed mesh showing iso-contours of effective stress



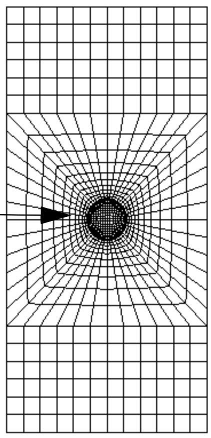
I. Cleavage fracture showing a microcrack growing across a grain boundary

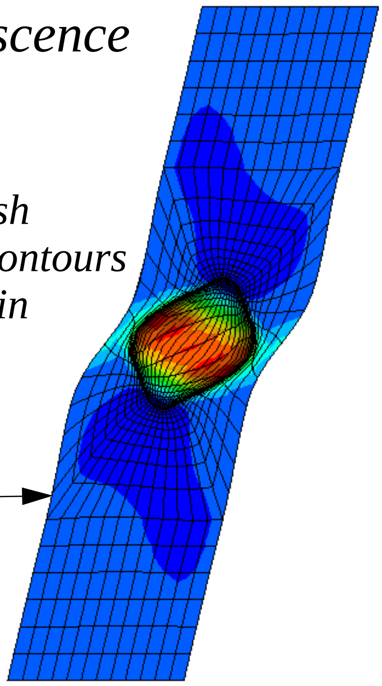
II. Ductile fracture by growth and coalescence of micovoids

Deformed mesh showing iso-contours of plastic strain

Undeformed FEM mesh

spherical void





FEM—historical aspects

1943

The method was outlined by the mathematician Richard Courant, but the method never caught the attention of engineers.

Mid 50s:

Developed and put to practical use on computers in the mid 50s by aeronautical structures engineers:

M.J. Turner, R.W. Clough, H.C. Martin, L.J. Topp (Boing and Bell Aerospace) in USA,

and

by J.H. Argyris and S. Kelsey (Rolls Royce) in UK.

1960s and later:

Theoretical basis for FEM was developed and mathematicians started to study different aspects of FEM, convergence, etc.

Development of FEM software began:

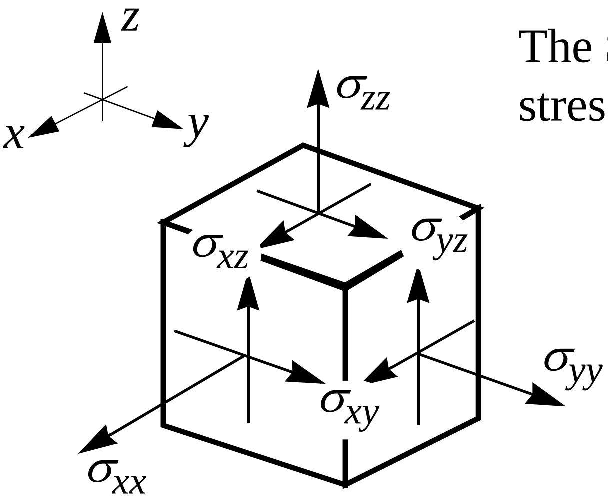
E. Wilson (freeware), D. MacNeal at NASA (general purpose program known as NASTRAN), J. Swanson at Westinghouse (developer of ANSYS), J. Hallquist at Livermore Nat. Lab. (LS-DYNA), ABAQUS developed by HKS (1978), and many many more ...

Recent development:

Various aspects of nonlinear problems, advanced material models, modelling of fracture, modelling of topology (automatic mesh generation), multi-physics (coupling of different physical phenomenon, e.g. fluid/structure interaction), and so on ...

But, the widespread use amongst engineers and scientist had never been possible without the ***exponential growth in the speed of computers*** and the even greater ***decline in the cost of computational resources!***

Stored elastic energy—multiaxial stress/strain states

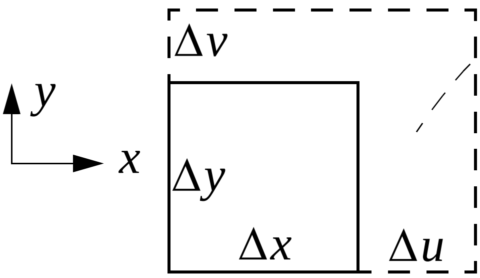


The **Stress matrix, S,** defines the stress state in a material point

$$\mathbf{S} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

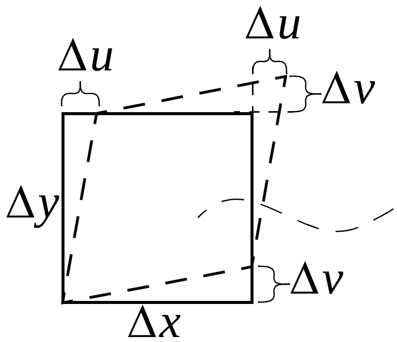
vector form

Normal strains (change of volume):



$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Shear strains (only change of shape):



$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

Stored in vector form: $\boldsymbol{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{xy} & \gamma_{xz} & \gamma_{yz} \end{bmatrix}$

Elastic strain energy / unit volume:

$$W' = \int(\sigma_{xx}\mathrm{d}\varepsilon_{xx} + \sigma_{yy}\mathrm{d}\varepsilon_{yy} + \sigma_{zz}\mathrm{d}\varepsilon_{zz} +$$

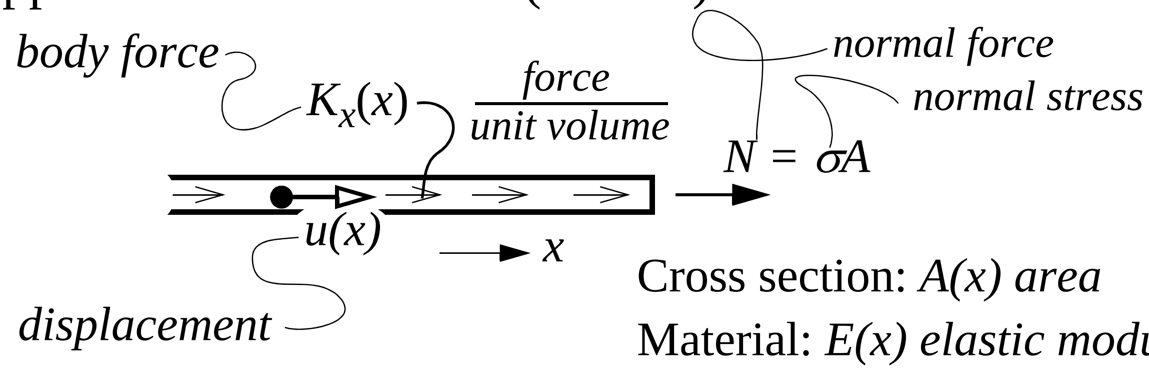
$$+ \sigma_{xy}\mathrm{d}\gamma_{xy} + \sigma_{xz}\mathrm{d}\gamma_{xz} + \sigma_{yz}\mathrm{d}\gamma_{yz})$$

For a linear elastisc material we obtain:

$$W' = \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz}) = \overline{W}'$$

Deformation in a Bar

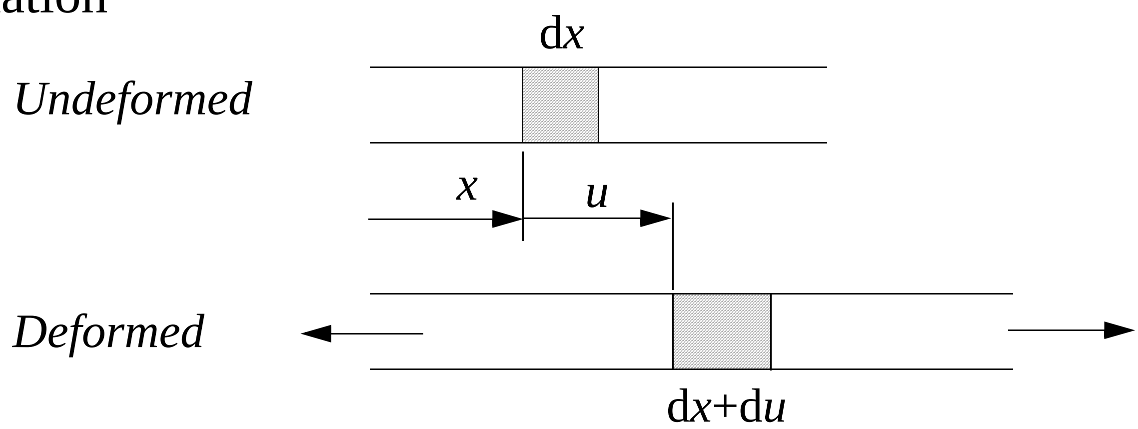
Applied forces & internal (section) forces



Cross section: *A(x) area*

Material: *E(x) elastic modulus*

Deformation



<u>Strain:</u>
(Compatibility)

$$\varepsilon = \frac{(\mathrm{d}x + \mathrm{d}u) - \mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} = u' \qquad (1)$$

<u>Constitutive Eq.:</u>
(Hooke's law)

$$N = \sigma A = \left\{ \begin{matrix} \text{Hooke} \\ \sigma = E\varepsilon \end{matrix} \right\} = EA\varepsilon = EAu' \qquad (2)$$

– –

<u>Equilibrium:</u>

$$\frac{dN}{dx} + AK_x = 0 \qquad (3)$$

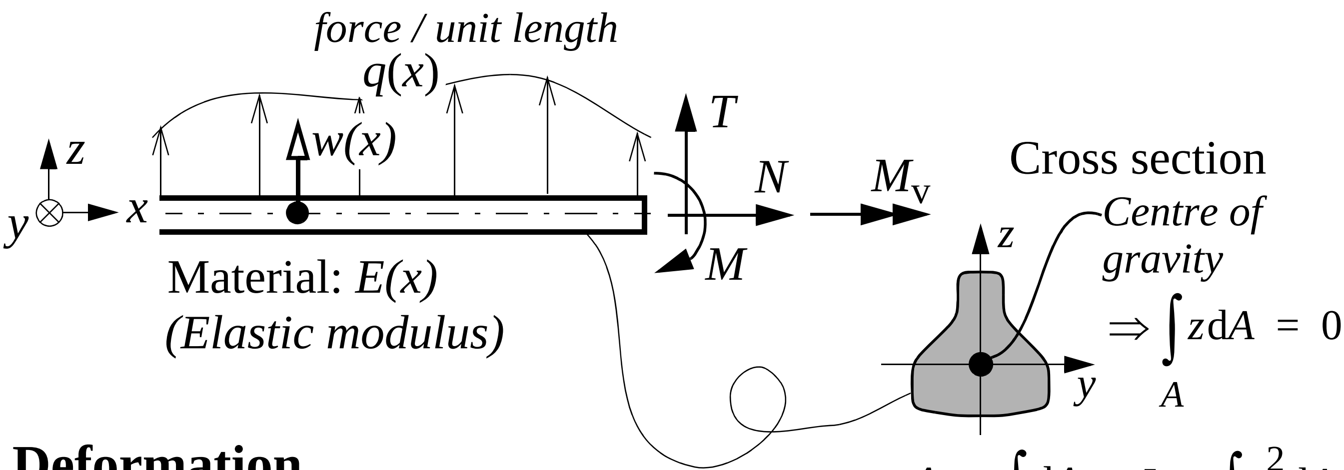
Eqs. (2) & (3) give:

$$\frac{d}{dx}EA\frac{du}{dx} + AK_x = 0$$

=> the solution to the diff. eq. gives the displacement *u(x)*

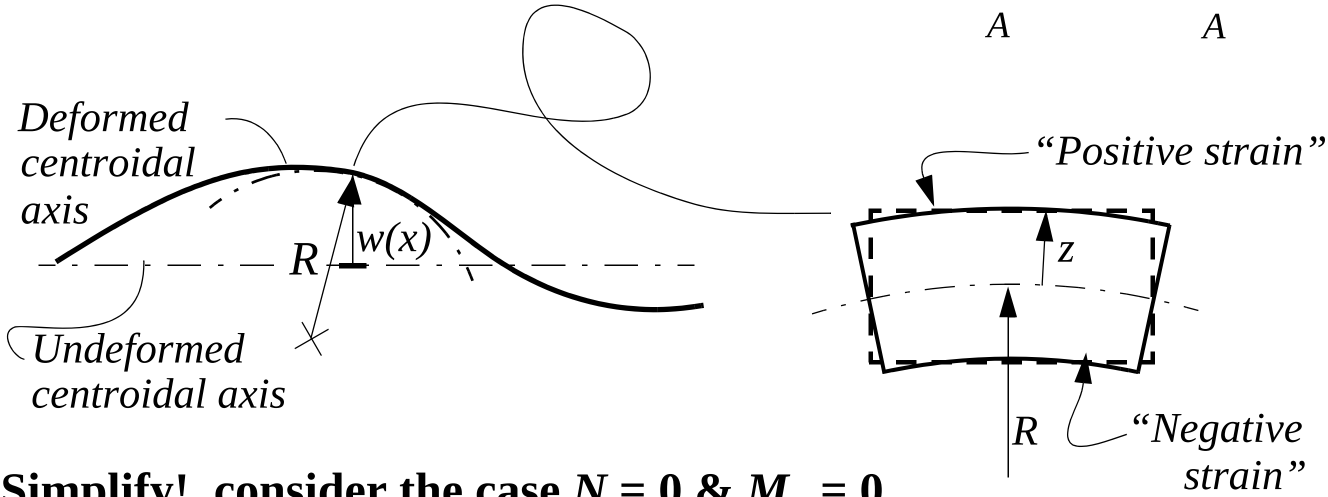
Deformation of a Beam

Applied forces & internal (section) forces



$$A = \int_A \mathrm{d}A \qquad I = \int_A z^2 \mathrm{d}A$$

Deformation



Simplify! consider the case $N = 0$ & $M_v = 0$

<u>Strain:</u>
(Compatibility)

$$\varepsilon = \frac{z}{R} = \left(\frac{-w''}{(1+(w')^2)^{3/2}}\right) \cdot z \approx -w''z \qquad (1)$$

<u>Constitutive Eq.:</u>
(Hooke's lag)

$$M = \int_A z\sigma \mathrm{d}A = \int_A zE(-w''z)\mathrm{d}A = -EIw'' \qquad (2)$$

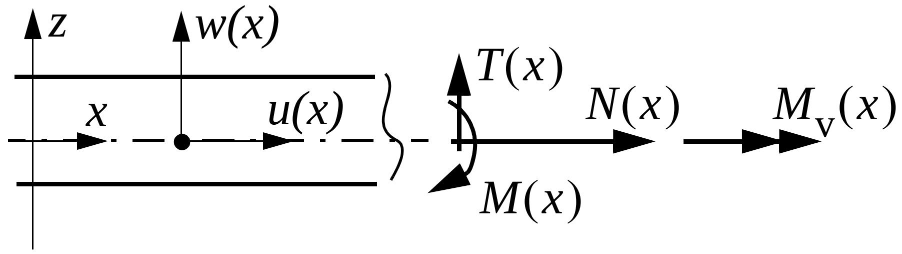
<u>Equilibrium:</u>

$$\frac{dM}{dx} = T, \qquad \frac{dT}{dx} = -q \qquad (3)$$

Eq. (2) & (3) give Euler-Bernoulli Eq. $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) - q = 0$

=> the solution to the diff. eq. gives the deflection *w(x)*

Elastic energy stored in a beam



Only considering normal strain (direction of the beam):

Constitutive relations:

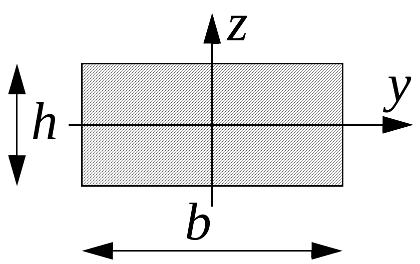
$$N = EAu'$$

$$M = -EIw''$$

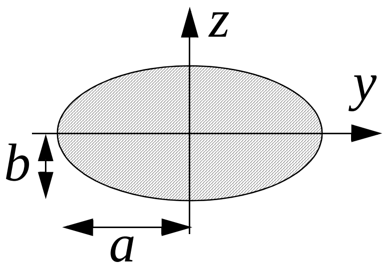
$$W = \int_L \left(\frac{EA}{2}(u')^2 + \frac{EI}{2}(w'')^2 \right) dx = \int_L \left(\frac{N^2}{2EA} + \frac{M^2}{2EI} \right) dx = \overline{W}$$

Accounting for shear strain (shear force & torque):

$$W = \overline{W} = \int_L \left[\frac{N^2}{2EA} + \frac{M^2}{2EI} + \beta \frac{T^2}{2GA} + \frac{M_v^2}{2GK} \right] dx$$



$$I_y = \frac{bh^3}{12} \quad \beta = 1{,}2 \quad K = \frac{bh^3}{3} f\left(\frac{b}{h}\right)$$



$$I_y = \frac{\pi a b^3}{4} \quad \beta = \frac{10}{9} \quad K = \frac{\pi a^3 b^3}{a^2 + b^2}$$

Lecture 2

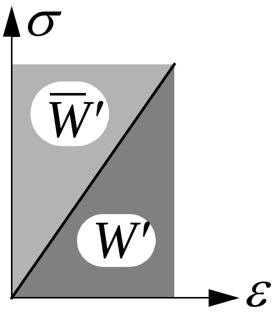
Rep. Work–Elastic Energy

"When an elastic solid deforms under the action of external forces, elastic energy is stored in the solid"

Example: linear elastic material

Point wise in the material elastic energy is stored as:

Uniaxial, $\sigma = E\varepsilon$



$$W' = \int \sigma \mathrm{d}\varepsilon = \frac{E\varepsilon^2}{2}$$

$$\overline{W}' = \int \varepsilon \mathrm{d}\sigma = \frac{\sigma^2}{2E}$$

$$\textit{where} \quad W' = \overline{W}' = \frac{1}{2}\sigma\varepsilon$$

Multiaxial, $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

$$W' = \overline{W}' = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})$$

Stresses and strains can therefore be expressed as:

Uniaxial: $\sigma = \dfrac{\partial W'}{\partial \varepsilon}$ $\qquad \varepsilon = \dfrac{\partial \overline{W}'}{\partial \sigma}$

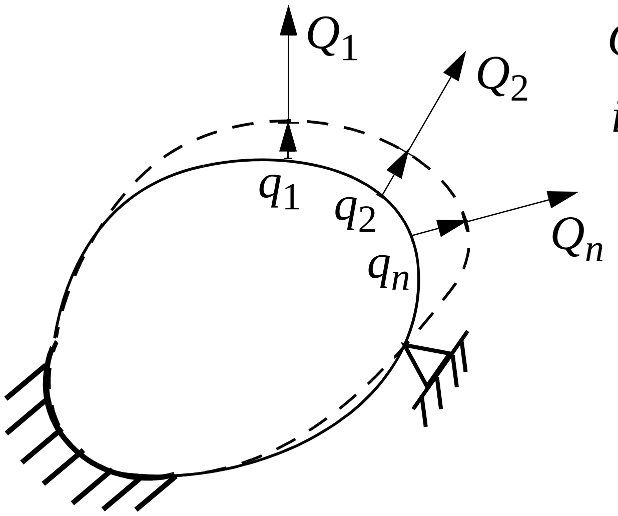
Multiaxial: $\sigma_{ij} = \dfrac{\partial W'}{\partial \varepsilon_{ij}}$ $\qquad \varepsilon_{ij} = \dfrac{\partial \overline{W}'}{\partial \sigma_{ij}}$

Total energy in the solid:

Elastic energy: $W = \int_V W' \mathrm{d}V$

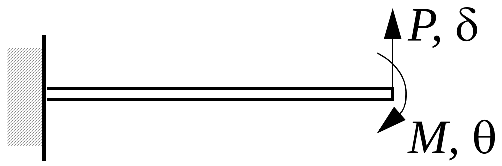
Complementary elastic energy: $\overline{W} = \int_V \overline{W}' \mathrm{d}V$

Discrete systems—Summary



Q_i and q_i are work conjugate,

i.e. that $Q_i \times q_i$ have units of work



$$[P \cdot \delta] = [\mathrm{Nm}]$$
$$[M \cdot \theta] = [\mathrm{Nm}]$$

Linear systems (linear elastic material & kinematics)

Flexibility form: $q_i = \sum_{j=1}^{n} \alpha_{ij} Q_j$	*Stiffness form*: $Q_i = \sum_{j=1}^{n} k_{ij} q_j$

Maxwell's reciprocal theorem: $\alpha_{ij} = \alpha_{ji}$ and $k_{ij} = k_{ji}$

(a_{ij} and k_{ij} are thus coefficients in **symmetric** matrices!)

Elastic energy: $W = \frac{1}{2}\sum_i \sum_i k_{ij} q_i q_j$

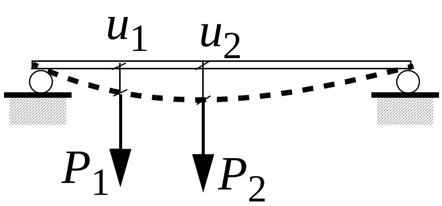
Castigliano's 1st theorem: $\frac{\partial W}{\partial q_i} = Q_i$

Complementary elastic energy: $\overline{W} = \frac{1}{2}\sum_i \sum_i \alpha_{ij} Q_i Q_j$

Castigliano's 2nd theorem: $\frac{\partial \overline{W}}{\partial Q_i} = q_i$

PHYSICAL INTERPRETATION OF MAXWELL'S RECIPROCAL THEOREM

$$\alpha_{ij} = \alpha_{ji} \qquad k_{ij} = k_{ji}$$

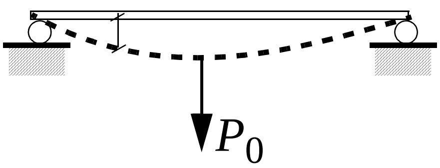


$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

Consider the two cases:

$P_1 = 0$ & $P_2 = P_0$

$$u_1 = \alpha_{12} P_0$$



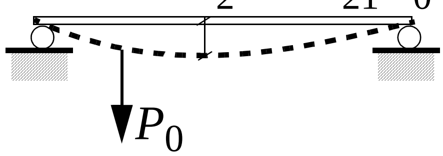
$$u_1 = u_2$$

since

$$\alpha_{12} = \alpha_{21}$$

$P_1 = P_0$ & $P_2 = 0$

$$u_2 = \alpha_{21} P_0$$



Note that!

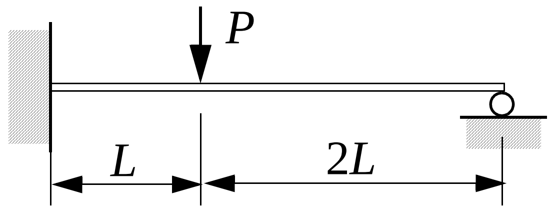
The coefficient α_{ij} determines the ***influence*** of the force P_j on the displacement u_i

Alternatively

The ***flexibility*** in the direction of displacement u_i due to the force P_j is determined by the coefficient α_{ij}

α_{ij} is therefore denoted *influence coefficient*
or *flexibility coefficient*

Statically indeterminate structures



Equilibrium:

$\uparrow$: $R_A + R_B - P = 0$

$\curvearrowright$ A: $-M_A + PL - R_B 3L = 0$

=> 3 unknowns – 2 Equil. Eqn. = 1 statically indeterminate, e.g. R_B

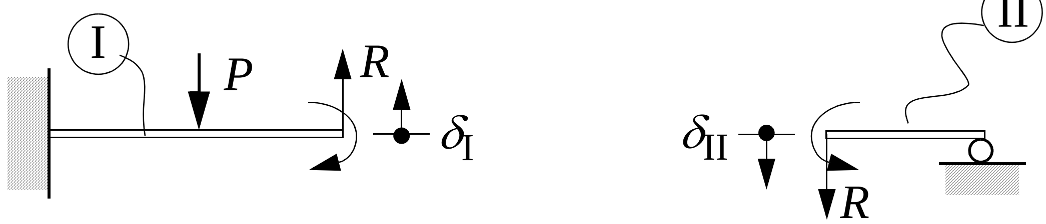
Solution:

$\delta_b = 0$ (kinematic constraint)

$$\delta_B = \frac{\partial \overline{W}}{\partial R_B} = 0 \quad \Rightarrow \quad \text{Equation to determine } R_B$$

Generalization—use an arbitrary internal force!

cut at x:



$$\overline{W}_I(P, R) \Rightarrow \delta_I = \frac{\partial \overline{W}_I}{\partial R} \qquad \overline{W}_{II}(P, R) \Rightarrow \delta_{II} = \frac{\partial \overline{W}_{II}}{\partial R}$$

but compatibility requires that $\delta_I = -\delta_{II} \quad \Leftrightarrow \quad \delta_I + \delta_{II} = 0$

$$\Rightarrow \delta_I + \delta_{II} = \frac{\partial \overline{W}_I}{\partial R} + \frac{\partial \overline{W}_{II}}{\partial R} = \frac{\partial}{\partial R}(\overline{W}_I + \overline{W}_{II}) = \frac{\partial}{\partial R}\overline{W} = 0$$

Thus: $\dfrac{\partial \overline{W}}{\partial R} = 0 \quad \Rightarrow \quad$ Equation to determine R

For the general case we obtain

Let $Q_1, ..., Q_n$ be n external generalized forces and

$R_1, ..., R_m$ be m statically indeterminate internal forces

$$\Rightarrow \overline{W} = \overline{W}(Q_1, \ldots, Q_n; R_1, \ldots, R_m)$$

then

$$\frac{\partial \overline{W}}{\partial R_k} = 0, \quad k = 1, \ldots, m \qquad \Rightarrow$$

m equations for the unknown $R_1, ..., R_m$

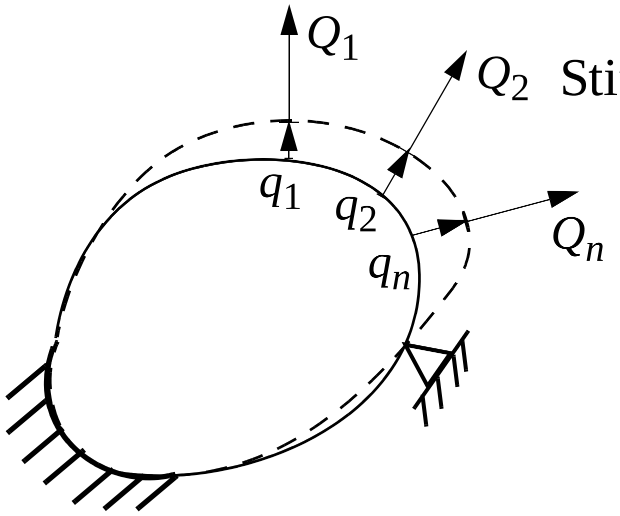
The solution takes the form: $\quad R_k = R_k(Q_1, \ldots, Q_n)$

Application of Castigliano's 2[nd] theorem then gives:

$$q_k = \frac{\partial \overline{W}}{\partial Q_k} = \sum_{l=1}^{m} \cancelto{0}{\frac{\partial \overline{W}}{\partial R_l}} \frac{\partial R_l}{\partial Q_k} + \frac{\partial \overline{W}}{\partial Q_k} = \left.\frac{\partial \overline{W}}{\partial Q_k}\right|_{R_l = \text{constant}}$$

Lecture 3 & 4

Matrix formulated Structure/Solid mechanics



Stiffness formulation $\sum_{j=1}^{n} k_{ij} q_j = Q_i$

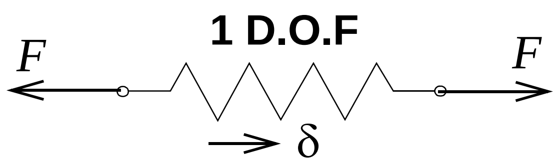
$$\underbrace{\begin{bmatrix} k_{11} & \cdots & k_{1n} \\ & \ddots & \vdots \\ \text{SYM.} & & k_{nn} \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix}}_{\mathbf{Q}}$$

How should the degrees of freedom (q_1, ..., q_n) be chosen and how should the stiffness matrix* K *be determined?

1. Divide the structure into elements
2. Use exact or approximate methods to describe the state in an element
3. The state in an element can often be described by a small number of degrees of freedom (D.O.F.)

Exemple:

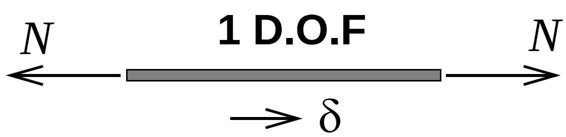
Spring



$F = k\delta$

stiffness

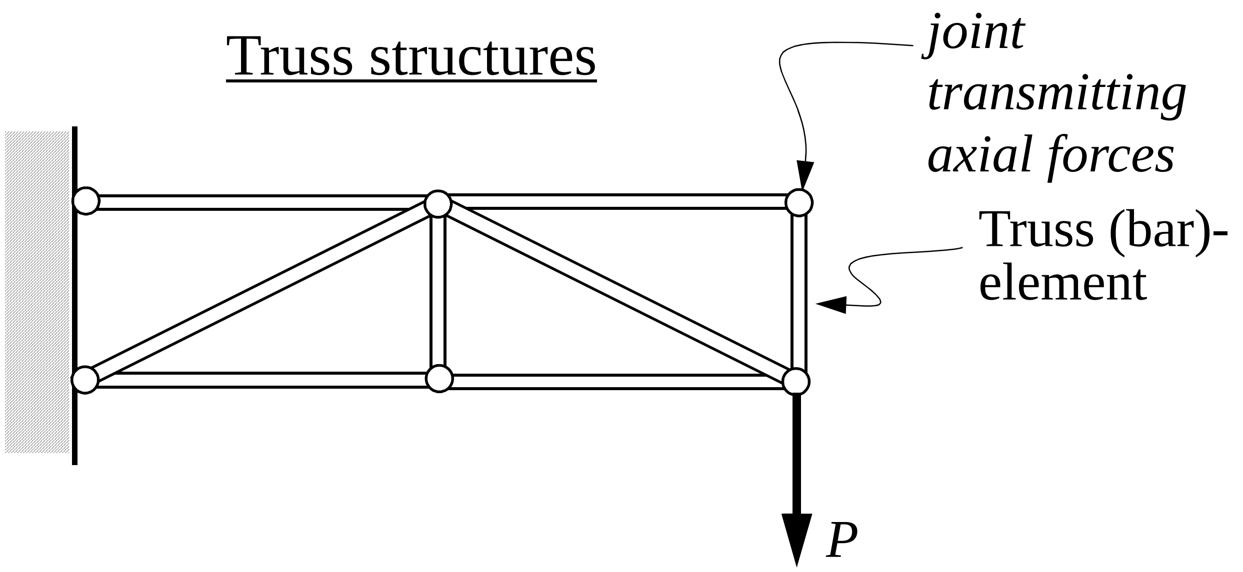
Truss/rod



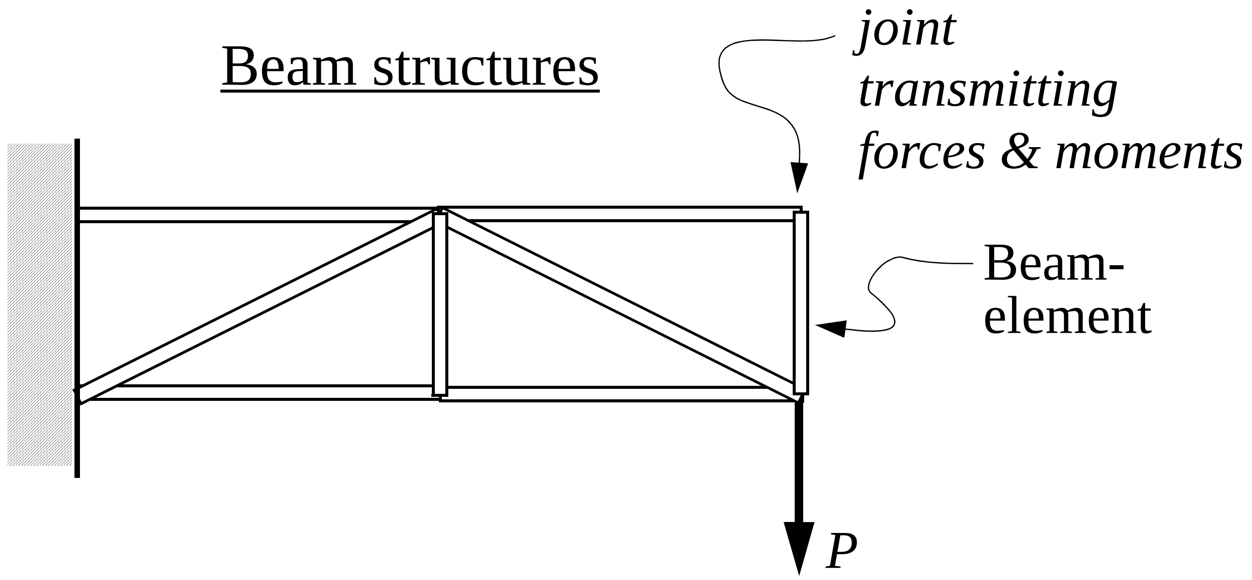
$N = \frac{EA}{L}\delta$

Structures — Solids

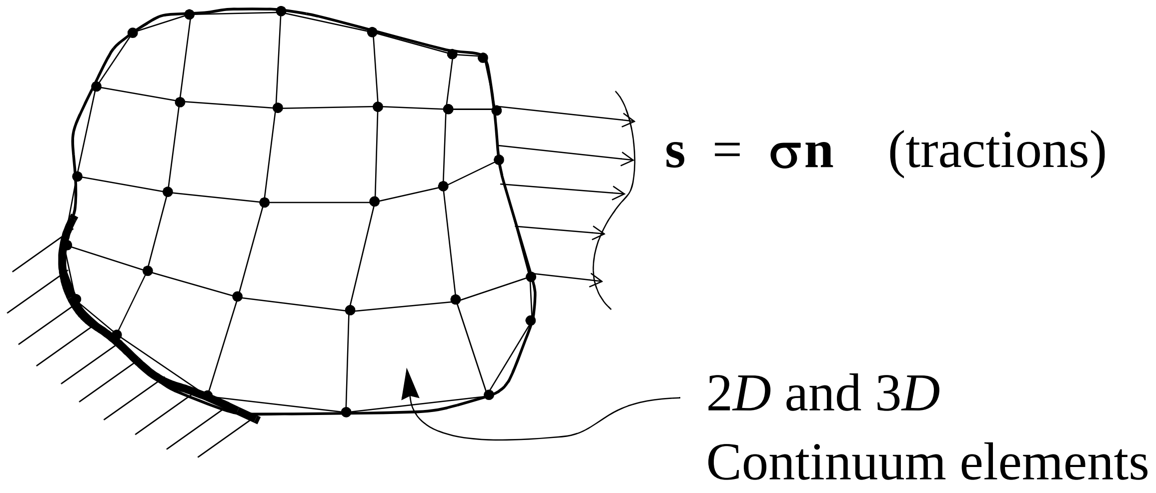
Truss structures



Beam structures

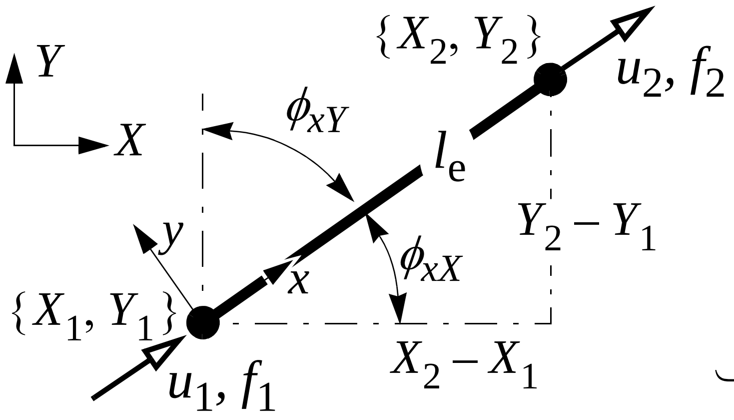


2- and 3-dimensional solids



One truss element in the plane (2D)

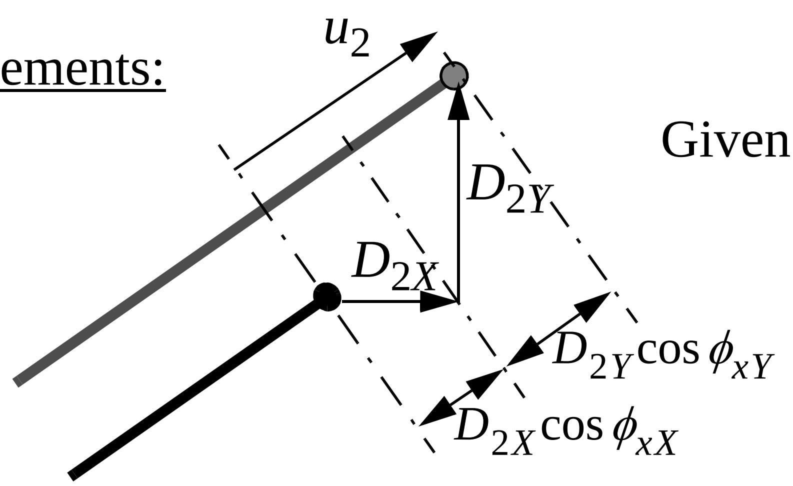
Local coord. syst. {x,y} —> Global coord. syst. {X,Y}



$$f_2 = -f_1 = k(u_2 - u_1)$$

$$\underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\mathbf{k}_e} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{d}_e} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{\mathbf{f}_e}$$

<u>Displacements:</u>



Given D_{2X} and D_{2Y}, determine u_2

$$\Rightarrow u_2 = D_{2X}\cos\phi_{xX} + D_{2Y}\cos\phi_{xY}$$

u_1 is determined in the same way!

"Direction cosines"

$$\Rightarrow \begin{cases} u_1 = D_{1X}\cos\phi_{xX} + D_{1Y}\cos\phi_{xY} \\ u_2 = D_{2X}\cos\phi_{xX} + D_{2Y}\cos\phi_{xY} \end{cases}$$

where $$\left.\begin{aligned} \cos\phi_{xX} &= (X_2 - X_1)/l_e = l_{12} \\ \cos\phi_{xY} &= (Y_2 - Y_1)/l_e = m_{12} \end{aligned}\right\}$$ *"Direction cosines"* where $l_{12}^2 + m_{12}^2 = 1$

$$l_e = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$

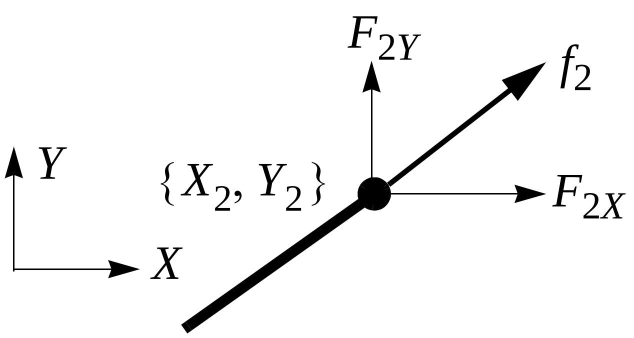
Matrix form:

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{d}_e} = \underbrace{\begin{bmatrix} l_{12} & m_{12} & 0 & 0 \\ 0 & 0 & l_{12} & m_{12} \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} D_{1X} \\ D_{1Y} \\ D_{2X} \\ D_{2Y} \end{bmatrix}}_{\mathbf{D}_e}$$

Transformation matrix

4 *D.O.F.*

<u>Nodal forces:</u>



Express the axial force, f_2, in components in the global coordinate system (project f_2 on the axis of coordinate syst.)

$$\Rightarrow \begin{cases} F_{2X} = f_2 \cos\phi_{xX} = f_2 l_{12} \\ F_{2Y} = f_2 \cos\phi_{xY} = f_2 m_{12} \end{cases}$$

Matrix form:

$$\underbrace{\begin{bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \end{bmatrix}}_{\mathbf{F}_e} = \underbrace{\begin{bmatrix} l_{12} & 0 \\ m_{12} & 0 \\ 0 & l_{12} \\ 0 & m_{12} \end{bmatrix}}_{\mathbf{T}^T} \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{\mathbf{f}_e}$$

Summary: local —> global transformation

Element stiffness matrix in ***local*** *coordinate system*

$$\left.\begin{array}{l}\left.\begin{array}{l}\mathbf{d}_e = \mathbf{T}\mathbf{D}_e \\ \mathbf{f}_e = \mathbf{k}_e\mathbf{d}_e\end{array}\right\} \Rightarrow \mathbf{f}_e = \mathbf{k}_e\mathbf{T}\mathbf{D}_e \\ \mathbf{F}_e = \mathbf{T}^T\mathbf{f}_e\end{array}\right\} \Rightarrow \mathbf{F}_e = \underbrace{\mathbf{T}^T\mathbf{k}_e\mathbf{T}}_{\mathbf{K}_e}\ \mathbf{D}_e$$

Element stiffness matrix in ***global*** *coordinate system*

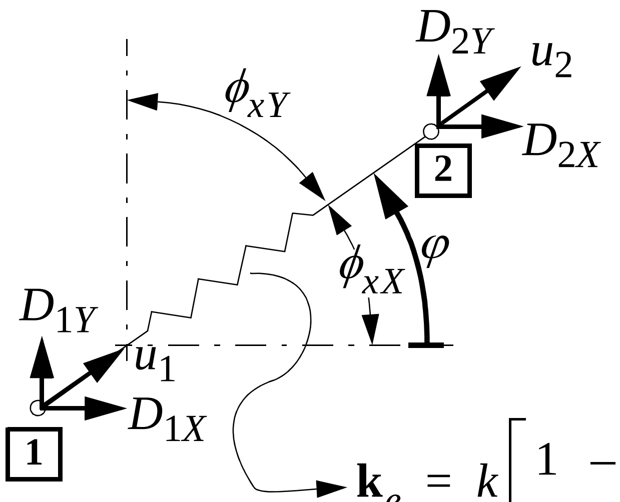
$$\Rightarrow \quad \boxed{\mathbf{F}_e = \mathbf{K}_e\ \mathbf{D}_e}$$

$$\text{where}\quad \mathbf{K}_e = \begin{bmatrix} l_{12} & 0 \\ m_{12} & 0 \\ 0 & l_{12} \\ 0 & m_{12} \end{bmatrix} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} l_{12} & m_{12} & 0 & 0 \\ 0 & 0 & l_{12} & m_{12} \end{bmatrix}$$

$$\Rightarrow \mathbf{K}_e = k\begin{bmatrix} \boldsymbol{a} & -\boldsymbol{a} \\ -\boldsymbol{a} & \boldsymbol{a} \end{bmatrix} \qquad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} l_{12}^2 & l_{12}m_{12} \\ l_{12}m_{12} & m_{12}^2 \end{bmatrix}$$

symmetric matrix!

Alternative formulation for the planar problem (2*D*)



$$\mathbf{k}_e = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$l_{12} = \cos\phi_{xX} = \cos\varphi = c$$

$$m_{12} = \cos\phi_{xY} = \cos\left(\frac{\pi}{2} - \varphi\right) = $$

$$= \sin\varphi = s$$

$$\Rightarrow T = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

Global element stiffness matrix in the plane:

$$\mathbf{K}_e = \mathbf{T}^T\mathbf{k}_e\mathbf{T} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} = k\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

The Global element stiffness matrix can also be derived by use energy methods (Castigliano's theorems)

Elastic energy:

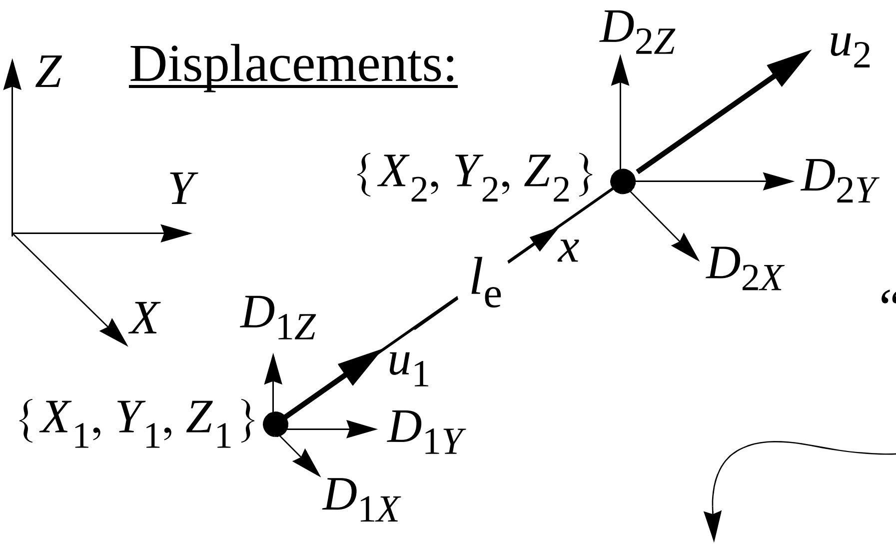
$$W = \frac{k}{2}(u_2 - u_1)^2 = \frac{k}{2}[\underbrace{(cD_{2X} + sD_{2Y})}_{u_2} - \underbrace{(cD_{1X} + sD_{1Y})}_{u_1}]^2$$

Castigliano's 1st theorem gives the components of the nodal forces as:

$$\left.\begin{aligned} F_{1X} &= \partial W/\partial D_{1x} \\ F_{1Y} &= \partial W/\partial D_{1y} \\ F_{2X} &= \partial W/\partial D_{2x} \\ F_{2Y} &= \partial W/\partial D_{2y} \end{aligned}\right\} \Rightarrow \underbrace{k\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}}_{\mathbf{K}_e}\underbrace{\begin{bmatrix} D_{1X} \\ D_{1Y} \\ D_{2X} \\ D_{2Y} \end{bmatrix}}_{\mathbf{D}_e} = \underbrace{\begin{bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \end{bmatrix}}_{\mathbf{F}_e}$$

One truss element in space (3*D*)

Local coord. syst. {x,y,z} —> Global coord. syst. {X,Y,Z}



"Direction cosines"

$$\Rightarrow \begin{cases} u_1 = D_{1X}\cos\phi_{xX} + D_{1Y}\cos\phi_{xY} + D_{1Z}\cos\phi_{zX} \\ u_2 = D_{2X}\cos\phi_{xX} + D_{2Y}\cos\phi_{xY} + D_{2Z}\cos\phi_{xZ} \end{cases}$$

where
$$\left.\begin{aligned} \cos\phi_{xX} &= (x_2 - x_1)/l_e = l_{12} \\ \cos\phi_{xY} &= (y_2 - y_1)/l_e = m_{12} \\ \cos\phi_{xZ} &= (z_2 - z_1)/l_e = n_{12} \end{aligned}\right\}$$

"Direction cosines" where

$$l_{12}^2 + m_{12}^2 + n_{12}^2 = 1$$

$$l_e = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

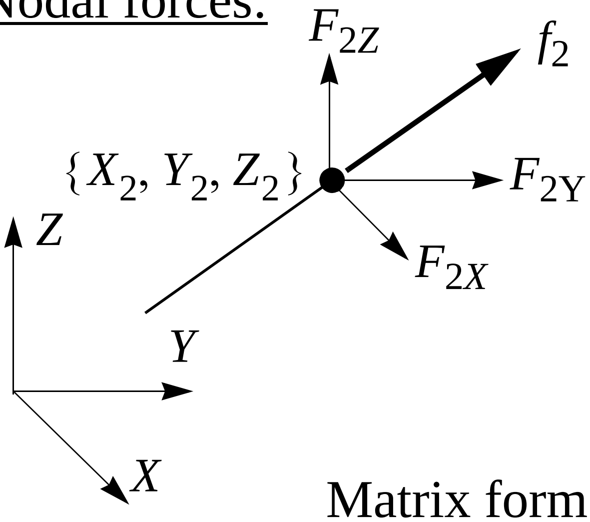
Matrix form:

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{d}_e} = \underbrace{\begin{bmatrix} l_{12} & m_{12} & n_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{12} & m_{12} & n_{12} \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} D_{1X} \\ D_{1Y} \\ D_{1Z} \\ D_{2X} \\ D_{2Y} \\ D_{2Z} \end{bmatrix}}_{\mathbf{D}_e}$$

Transformation matrix

6 D.O.F.

<u>Nodal forces:</u>



$$\Rightarrow \begin{cases} F_{2X} = f_2 \cos\phi_{xX} = f_2 l_{12} \\ F_{2Y} = f_2 \cos\phi_{xY} = f_2 m_{12} \\ F_{2Z} = f_2 \cos\phi_{xZ} = f_2 n_{12} \end{cases}$$

Matrix form:

$$\underbrace{\begin{bmatrix} F_{1X} \\ F_{1Y} \\ F_{1Z} \\ F_{2X} \\ F_{2Y} \\ F_{2Z} \end{bmatrix}}_{\mathbf{F}_e} = \underbrace{\begin{bmatrix} l_{12} & 0 \\ m_{12} & 0 \\ n_{12} & 0 \\ 0 & l_{12} \\ 0 & m_{12} \\ 0 & n_{12} \end{bmatrix}}_{\mathbf{T}^T} \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{\mathbf{f}_e}$$

Summary: local—>global transformation in 3D

Element stiffness matrix in ***local*** *coordinate system*

$$\left.\begin{array}{l} \left.\begin{array}{l} \mathbf{d}_e = \mathbf{T}\mathbf{D}_e \\ \mathbf{f}_e = \mathbf{k}_e\mathbf{d}_e \end{array}\right\} \Rightarrow \mathbf{f}_e = \mathbf{k}_e\mathbf{T}\mathbf{D}_e \\ \mathbf{F}_e = \mathbf{T}^T\mathbf{f}_e \end{array}\right\} \Rightarrow \mathbf{F}_e = \underbrace{\mathbf{T}^T\mathbf{k}_e\mathbf{T}}_{\mathbf{K}_e}\ \mathbf{D}_e$$

Element stiffness matrix in ***global*** *coordinate system*

$$\Rightarrow \boxed{\mathbf{F}_e = \mathbf{K}_e\ \mathbf{D}_e}$$

Here

$$\mathbf{K}_e = \mathbf{T}^T\mathbf{k}_e\mathbf{T} = k\begin{bmatrix} \mathbf{a} & -\mathbf{a} \\ -\mathbf{a} & \mathbf{a} \end{bmatrix} \quad \text{where} \quad \mathbf{a} = \begin{bmatrix} l_{12}^2 & l_{12}m_{12} & l_{12}n_{12} \\ l_{12}m_{12} & m_{12}^2 & m_{12}n_{12} \\ l_{12}n_{12} & m_{12}n_{12} & n_{12}^2 \end{bmatrix}$$

symmetric matrix!

Numbering of the Equations

- Total number of equations (= D.O.F.) are equal to

$$[\text{D.O.F. per node}] \times N$$

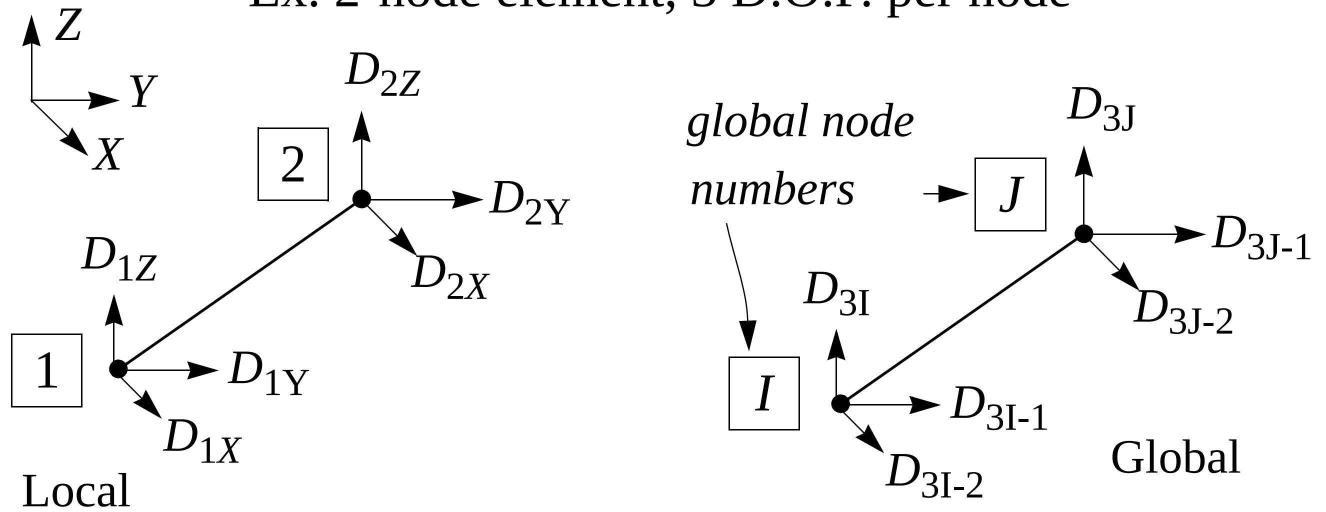
total number of nodes in the model

- Numerical analysis requires systematic numbering of Eqs. (e.g. to handle boundary conditions etc.)

- Assembly of global stiffness matrix and load vector also requires relation between local and global numbering of D.O.F.

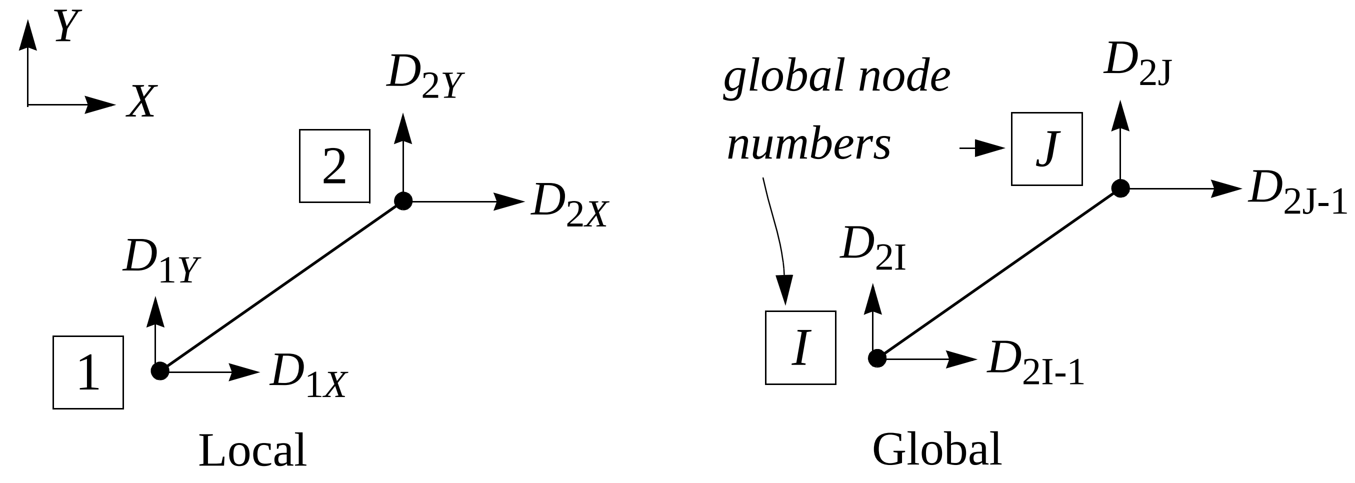
$\Rightarrow$ Book keeping problem!

Ex. 2-node element, 3 D.O.F. per node

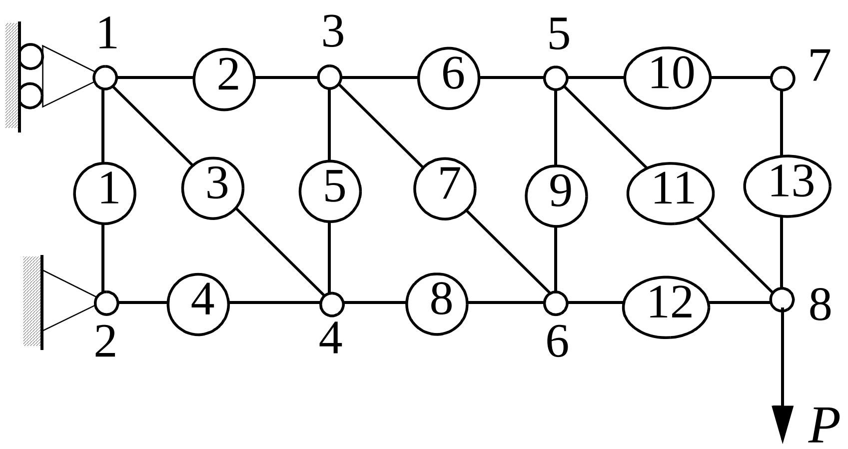


Degrees of freedom		Eq. number
Local	Global	(ex. I=1 & J=8)
D_{1X}	$D_{3\mathrm{I}-2}$	1
D_{1Y}	$D_{3\mathrm{I}-1}$	2
D_{1Z}	$D_{3\mathrm{I}}$	3
D_{2X}	$D_{3\mathrm{J}-2}$	22
D_{2Y}	$D_{3\mathrm{J}-1}$	23
D_{2Z}	$D_{3\mathrm{J}}$	24

Ex. 2-node element, 2 D.O.F. per node

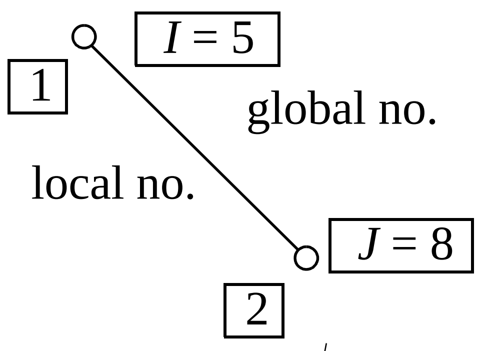


Ex.: a planar truss structure (trusses/rods or spring elements)



13 elements
8 nodes => 16 D.O.F.

Element no. 11



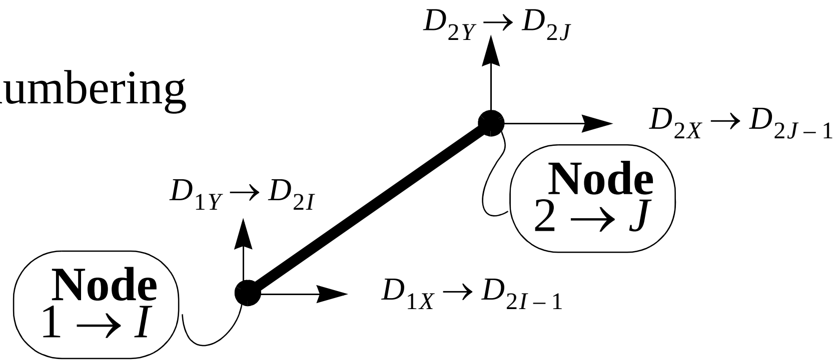
Degrees of freedom Local	Global	Eq. number (e.g. I=5 & J=8)
D_{1X}	D_{2I-1}	9
D_{1Y}	D_{2I}	10
D_{2X}	D_{2J-1}	15
D_{2Y}	D_{2J}	16

Boundary
Conditions: $D_1 = D_3 = D_4 = 0$ (F_1, F_3 and F_4 reaction forces)

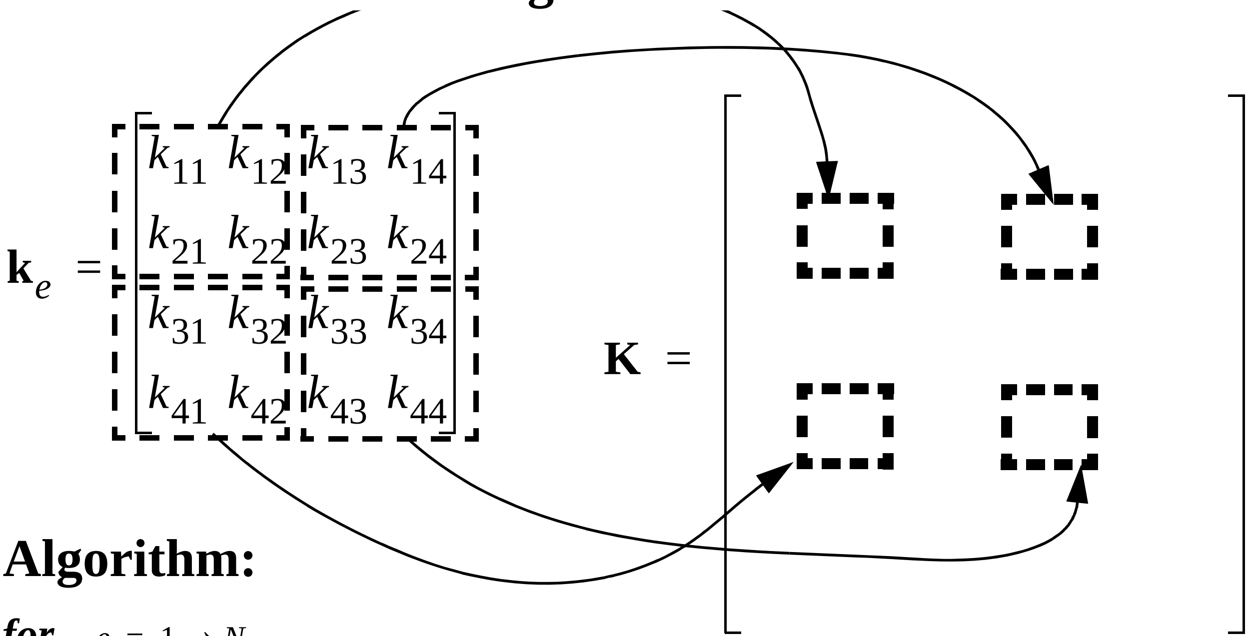
$$F_2 = F_5 = F_6 = \ldots = F_{15} = 0, \quad F_{16} = -P$$

Algorithm for assembly of global stiffness matrix

Local $\rightarrow$ Global numbering



Assembly: element stiffness matrices are added to the global stiffness matrix



Algorithm:

for $e = 1 \rightarrow N_{el}$

Compute the element stiffness matrix $\mathbf{k}_e$

$$\left.\begin{aligned} \text{Ekv}(1) &= 2I-1 \\ \text{Ekv}(2) &= 2I \\ \text{Ekv}(3) &= 2J-1 \\ \text{Ekv}(4) &= 2J \end{aligned}\right\} \text{ Equation numbers are determined by the element node numbers}$$

for $i = 1 \rightarrow 4$

for $j = 1 \rightarrow 4$

add $\mathbf{k}_e$ *to* $\mathbf{K}$

$$\mathbf{K}(\text{ Ekv}(i),\ \text{Ekv}(j)\ ) = \mathbf{K}(\text{ Ekv}(i),\ \text{Ekv}(j)\ ) + \mathbf{k}_e(i,j)$$

row *column*

end

end

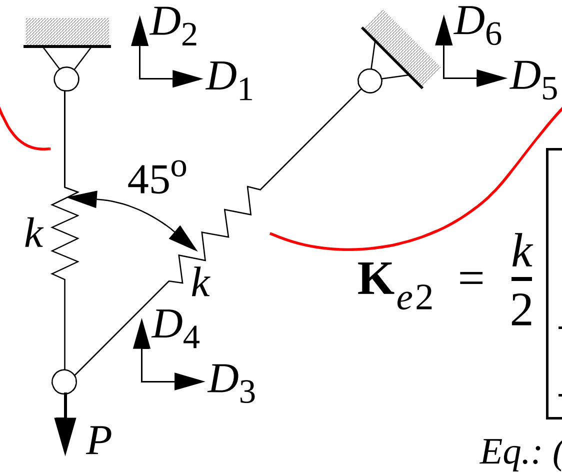
end

See the Matlab program: **spring2D** & **truss2D** on the home page!

Truss structure example—Summary

$$\mathbf{K}_{e1} = \frac{k}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}$$

Eq.: (1) (2) (3) (4)



$$\mathbf{K}_{e2} = \frac{k}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

Eq.: (3) (4) (5) (6)

Boundary Conditions: $D_1 = D_2 = D_5 = D_6 = 0$; $F_3 = 0$, $F_4 = -P$

Equation system

$$\frac{k}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & -2 & 1 & 3 & -1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ 0 \\ -P \\ R_5 \\ R_6 \end{bmatrix}$$

Unknown

Known

Computational steps:

1. Calculate *element stiffness matrices* and **assemble** *global stiffness matrix*

2. Solve for the unknown *displacements* (Eqs. 3, 4) = > D_3, D_4

$$\frac{k}{2}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} = \frac{P}{k}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3. Calculate the unknown *reaction forces* (Eqs. 1,2, 5,6)

Eq. (2): $R_2 = k/2(2D_2 - 2D_4) = P$

Eq. (5): $R_5 = k/2(-D_3 - D_4 + D_5 + D_6) = 0$

Eq. (6): $\Rightarrow R_6 = 0$

Lectures 5, 6 and 7

Introduction to *approximate* solution methods in solid mechanics

1. Principle of Virtual Work (PVW)

2. Approximate methods based on PVW

3. General method for development of FEM-Eq. based on the weak form (a generalization of PVW, applicable to PDE:s in general)

4. Procedure for FEM-analysis with application to uniaxial problems (trusses and planar truss structures)

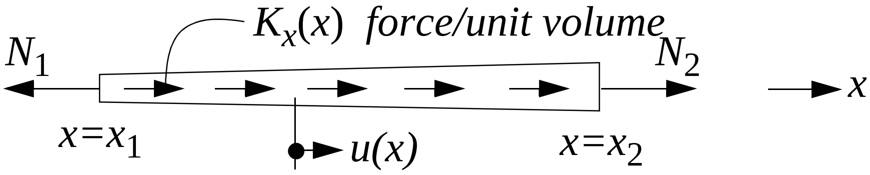
Principle of virtual work

at equilibrium holds $\delta A^{(e)} = \delta A^{(i)}$

virtual work of external forces — *virtual work of internal forces*

"Necessary and sufficient condition for equilibrium"

Uniaxial application (bar):



<u>Equilibrium:</u> $\dfrac{dN}{dx} + AK_x = 0$ <u>Compatibility:</u> $\varepsilon = \dfrac{\mathrm{d}u}{\mathrm{d}x}$

Introduce an **arbitrary variation in displacement** $\delta u(x)$ from the equilibrium pos. with a **compatible variation in strain** $\delta\varepsilon = \mathrm{d}\delta u/\mathrm{d}x$
$u(x) + \delta u(x)$ must satisfy **geometrical boundary cond. & constraint**.

Thus, <u>$\delta u(x) = 0$ where $u(x)$ is prescribed</u>

External forces $\{N_1, N_2$ & $K_x\}$ then perform the work

$$\delta A^{(e)} = \underbrace{N_2\delta u(x_2) + N_1(-\delta u(x_1))}_{} + \int_{x_1}^{x_2} \delta u K_x A\,\mathrm{d}x$$

$$[N\delta u]_{x_1}^{x_2} = \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}[N\delta u]\mathrm{d}x = \int_{x_1}^{x_2} \left[\frac{dN}{\mathrm{d}x}\delta u + N\underbrace{\frac{\mathrm{d}\delta u}{\mathrm{d}x}}_{\delta\varepsilon}\right]\mathrm{d}x$$

$$\Rightarrow \delta A^{(e)} = \int_{x_1}^{x_2} \underbrace{\left(\frac{dN}{\mathrm{d}x} + K_x A\right)}_{= 0 \text{ due to equilibrium!}} \delta u\,\mathrm{d}x + \int_{x_1}^{x_2} \underbrace{N}_{\sigma A}\,\delta\varepsilon\,\mathrm{d}x$$

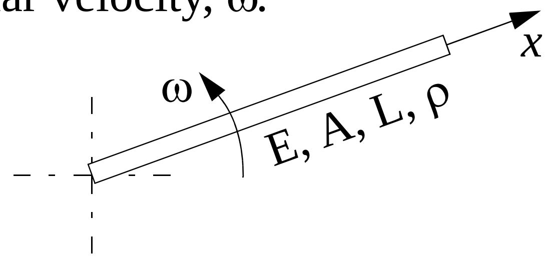
Thus, $\delta A^{(e)} = \int_{x_1}^{x_2} \sigma\delta\varepsilon A dx = \int_V \underbrace{\sigma\,\delta\varepsilon}_{} \,\mathrm{d}V = \delta A^{(i)}$

internal virtual work / unit volume

Note! this is valid regardless the material behaviour!

Illustration of virtual displacement in P.V.W

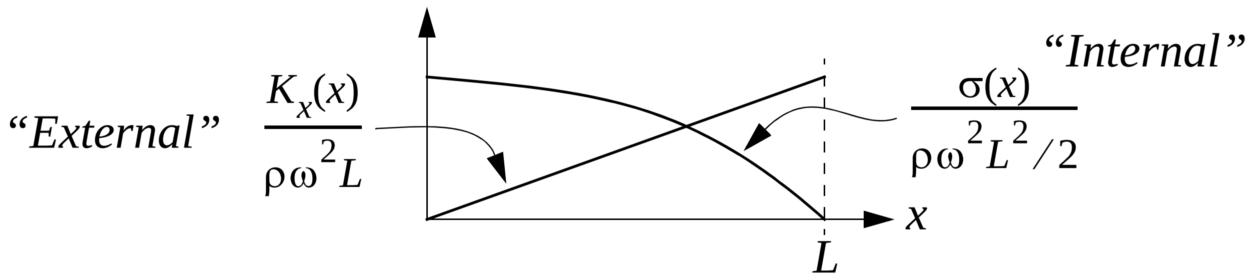
Example: Truss, rotating at a constant angular velocity, ω.



The load, centrifugal force, is introduced as a body force

$$K_x = \rho\omega^2 x$$

$$\left.\begin{aligned} &\underline{\text{Equilibrium:}} \quad \frac{d}{dx}(\sigma A) + K_x A = 0 \\ &\underline{\text{B.C.:}} \quad \sigma(x = L) = 0 \end{aligned}\right\} \Rightarrow \sigma(x) = \frac{\rho\omega^2 L^2}{2}\left(1 - \left(\frac{x}{L}\right)^2\right)$$



$$\left.\begin{aligned} &\underline{\text{Compatibility:}} \quad \frac{du}{dx} = \varepsilon = \frac{\sigma}{E} \\ &\underline{\text{B.C.:}} \quad u(x = 0) = 0 \end{aligned}\right\} \Rightarrow u(x) = \frac{\rho\omega^2 L^3}{6E}\frac{x}{L}\left(3 - \left(\frac{x}{L}\right)^2\right)$$

Study a displacement variation $\delta u(x)$ (virtual displacement), around $u(x)$, given as $\delta u = \alpha\sin(\beta\pi x/L)$, $\quad \alpha, \beta > 0$

$$\Rightarrow \delta\varepsilon = \frac{\alpha\beta\pi}{L}\cos(\beta\pi x/L)$$

Internal virtual work:

$$\delta A^{(i)} = \int_0^L \delta\varepsilon(x)\sigma(x)A\,dx = \ldots = AL^2\rho\omega^2\alpha\frac{(\sin\beta\pi - \beta\pi\cos\beta\pi)}{\beta^2\pi}$$

External virtual work:

$$\delta A^{(e)} = \int_0^L \delta u(x)K_x(x)A\,dx = \ldots = AL^2\rho\omega^2\alpha\frac{(\sin\beta\pi - \beta\pi\cos\beta\pi)}{\beta^2\pi}$$

Thus, $\delta A^{(i)} = \delta A^{(e)}$, independent of α and β as stated by ***P.V.W.***

Approximate solution method based on the Principle of Virtual Work

General features:

(i) **Compatibility** and **material relation** will be satisfied everywhere!

(ii) **Equilibrium** will not be satisfied everywhere, only in an average sense!

Computational steps (truss example):

1. Make an ***approximate ansatz*** *(trial function)*, $\tilde{u}(x)$, for the displacement solution.

Requirements: $\tilde{u}(x)$ must satisfy ***kinematic boundary conditions & constraints***.

A rather ***general ansatz*** is:

A priori unknown coefficients to be determined

$$\tilde{u}(x) = \phi_0(x) + \sum_{j=1}^{n} \alpha_j \phi_j(x)$$

Fulfils kinematic B.C & constraints

"Basis functions" equal to zero at boundaries with kinematic B.C.

2. Determine α_i by use of the Principle of Virtual Work (P.V.W.). A convenient choice for the ***displacement variation*** *(test function)* $\delta u(x)$ is:

$$\delta u(x) = \sum_{i=1}^{n} \beta_i \phi_i(x), \quad \beta_i \text{ are } \textbf{\textit{arbitrary coefficients.}}$$

$$\Rightarrow \delta\varepsilon(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = \sum_{i=1}^{n} \beta_i \phi'_i(x) \text{ (compatible virtual strain)}$$

Internal virtual work

$$\delta A^{(i)} = \int_{x_1}^{x_2} \delta\varepsilon(x)\tilde{\sigma}(x)A\,\mathrm{d}x = \left\{\tilde{\sigma} = E\frac{d\tilde{u}}{dx}\right\} =$$

$$= \sum_{i=1}^{n} \beta_i \left[\int_{x_1}^{x_2} \phi'_i EA\left(\phi'_0(x) + \sum_{j=1}^{n} \alpha_j \phi'_j(x)\right)\mathrm{d}x\right]$$

External virtual work

$$\delta A^{(e)} = [\delta u(x)N]_{x_1}^{x_2} + \int_{x_1}^{x_2} \delta u(x)K_x(x)A\,\mathrm{d}x =$$

$$= \sum_{i=1}^{n} \beta_i \left[[\phi_i N]_{x_1}^{x_2} + \int_{x_1}^{x_2} \phi_i K_x(x)A\,\mathrm{d}x\right]$$

Now, invoke P.V.W., which states that $\delta A^{(i)} = \delta A^{(e)}$ should be satisfied for arbitrary choices of β_i. Hence, we obtain a system of n equations for the n unknown coefficients α_j.

$$\int_{x_1}^{x_2} \phi'_i EA\left(\phi'_0(x) + \sum_{j=1}^{n} \alpha_j \phi'_j(x)\right)\mathrm{d}x = [\phi_i N]_{x_1}^{x_2} + \int_{x_1}^{x_2} \phi_i K_x(x)A\,\mathrm{d}x$$

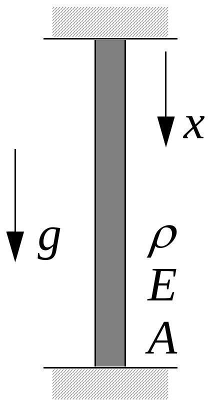
$$i = 1, \ldots, n$$

On ***matrix form*** this reads

$$\begin{bmatrix} A_{11} & \ldots & A_{1n} \\ & & \\ A_{n1} & \ldots & A_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ b_n \end{bmatrix} \Rightarrow \quad \text{solve for } \boldsymbol{\alpha}$$

$$\boxed{\int_{x_1}^{x_2} \phi'_n EA \phi'_1\,\mathrm{d}x} \qquad \boxed{[\phi_n N]_{x_1}^{x_2} + \int_{x_1}^{x_2} \phi_n K_x A\,\mathrm{d}x - \int_{x_1}^{x_2} \phi'_n EA \phi'_0\,\mathrm{d}x}$$

Uniaxial example: truss with axial load



A linear elastic bar (E) is loaded by its dead weight $K_x = \rho g$. Determine the displacement in the bar with an approximate methods based on the Principle of virtual work (P.V.W.)

Boundary conditions: $u(x=0) = 0$ & $u(x=L) = 0$

P.V.W.: $$\delta A^{(i)} = \int_{x_1}^{x_2} \delta\varepsilon\sigma A dx = [N\delta u]_{x_1}^{x_2} + \int_{x_1}^{x_2} \delta u K_x A \, \mathrm{d}x = \delta A^{(e)}$$

Approximate ansatz: $$\tilde{u}(x) = c_0 + c_1\frac{x}{L} + c_2\left(\frac{x}{L}\right)^2$$

with B.C.: $u(0) = u(L) = 0 \quad \Rightarrow \quad c_0 = 0, c_2 = -c_1$

we obtain, $$\tilde{u}(x) = c_1\frac{x}{L}\left(1-\frac{x}{L}\right) \quad \Rightarrow \quad \tilde{\varepsilon}(x) = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} = \frac{c_1}{L}\left(1-2\frac{x}{L}\right)$$

Choice of δu: $$\delta u = d\frac{x}{L}\left(1-\frac{x}{L}\right) \Rightarrow \delta\varepsilon = \frac{\mathrm{d}\delta u}{\mathrm{d}x} = \frac{d}{L}\left(1-2\frac{x}{L}\right)$$

Hooke's law: $\sigma = E\,\tilde{\varepsilon}(x) = E\,\dfrac{\mathrm{d}\tilde{u}(x)}{\mathrm{d}x}$ **Note! the ansatz is used here!**

Solution:

$$\delta A^{(i)} = \int_{x_1}^{x_2} \delta\varepsilon\sigma A dx = \int_{x_1}^{x_2} \frac{d}{L}\left(1-2\frac{x}{L}\right)E\frac{c_1}{L}\left(1-2\frac{x}{L}\right)A dx = d\frac{EA}{3L}c_1$$

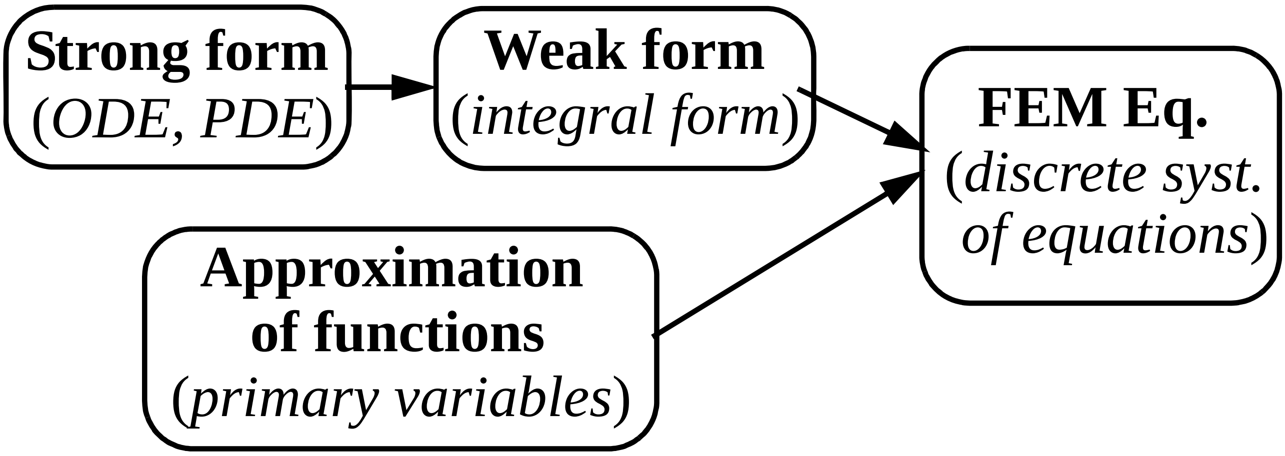
$$\delta A^{(e)} = 0 + \int_{x_1}^{x_2} d\frac{x}{L}\left(1-\frac{x}{L}\right)\rho g A \, \mathrm{d}x = d\frac{\rho g A L}{6}$$

$$\delta A^{(i)} = \delta A^{(e)} \Rightarrow d\left(\frac{EA}{3L}c_1 - \frac{\rho g A L}{6}\right) = 0 \Rightarrow c_1 = \frac{\rho g L^2}{2E}$$

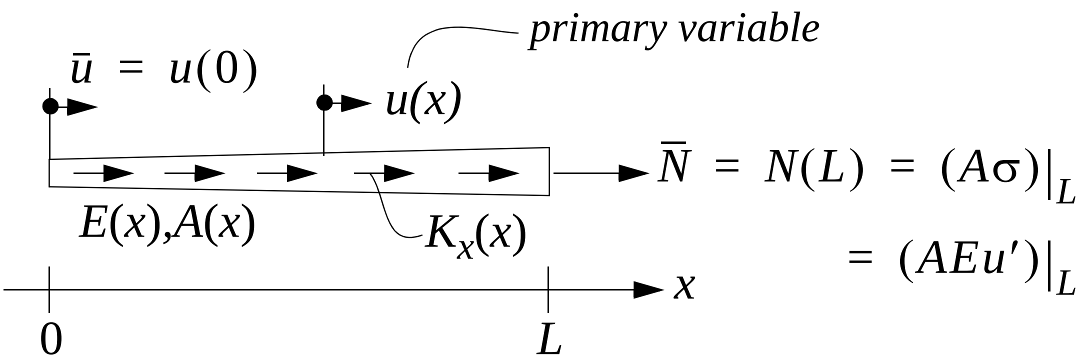
$$\Rightarrow \tilde{u}(x) = \frac{\rho g L^2}{2E}\frac{x}{L}\left(1-\frac{x}{L}\right)$$ **The Exact solution in this case!**

Development of FEM-Equations

— General procedure for physical problems described by a PDE



<u>Example: Truss (1D)</u>



Strong form:

<u>O.D.E.:</u>

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) + K_x A = 0 \quad \text{för} \quad 0 < x < L$$

<u>Boundary Conditions:</u>

$x = 0: \quad u = \bar{u}$ (essential)

$x = L: \quad AEu' = \bar{N}$ (natural)

Weak form (integral form, variational form):

1. Multiply **O.D.E.** and **B.C.** by an ***arbitrary weight function***, *v(x)*, and integrate over the length of the truss:

$$\Rightarrow \begin{cases} \displaystyle\int_0^L v(x)\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) + K_x A\right]\mathrm{d}x = 0 & \text{(1a)} \\ [v(x)(\bar{N} - AEu')]\big|_{x = L} = 0 & \text{(1b)} \\ \text{Suitable restriction for } v(x)\text{ , choose } v(0) = 0 & \text{(1c)} \end{cases}$$

2. Integrate the 1st term in (1a) by parts, i.e. lower u'' to u':

$$\Rightarrow \int_0^L v(x)\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right]\mathrm{d}x = \left[v(x)EA\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L - \int_0^L \frac{\mathrm{d}v}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

inserted into (1a) gives

$$\int_0^L \frac{\mathrm{d}v}{\mathrm{d}x}E\frac{\mathrm{d}u}{\mathrm{d}x}A\mathrm{d}x = \underbrace{v(EAu')\big|_{x = L}}_{= \bar{N}\ \text{(1b)}} - \underbrace{v(EAu')\big|_{x = 0}}_{= 0\ \text{(1c)}} + \int_0^L vK_xA\mathrm{d}x$$

Weak form, definition: Find *u(x)* among all admissible functions that satisfies the essential B.C. ($u(0) = \bar{u}$), such that

$$\int_0^L \frac{\mathrm{d}v}{\mathrm{d}x}E\frac{\mathrm{d}u}{\mathrm{d}x}A\mathrm{d}x - \left[(v\bar{N})\big|_{x = L} + \int_0^L vK_xA\mathrm{d}x\right] = 0$$

for an arbitrary $v(x)$ with $v(0) = 0$

WEAK FORM ⇔ STRONG FORM

PHYSICAL INTERPRETATION = PRINCIPLE OF VIRTUAL WORK

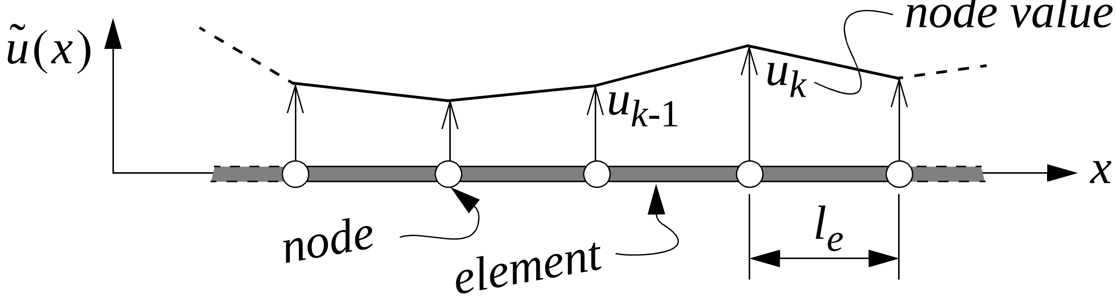
FEM—Approximate solution of weak form

The discretized system of FEM-equations results after choice of

- ***Approximate solution ansatz*** $\tilde{u}(x)$ (trial function)
- ***Weight function*** $v(x)$ (test function)

Piece wise continues functions are used in **FEM**, i.e. the geometry is divided into ***elements*** connected by ***nodes***.

eg. piece wise linear fcn.



$\tilde{u}(x)$ and $v(x)$ must satisfy the conditions:

(i) ***Continuity*** across element boundaries,

(ii) ***Completeness***, i.e. the functions themselves and their derivatives up to highest order appearing in the weak form must be capable of assuming constant values.

(i) and (ii) are necessary conditions for convergence

$$\tilde{u}(x) \to u(x) \quad \text{when} \quad l_e \to 0$$

Exemples on completeness in 1D:

$$\tilde{u} = c_0 + c_1 x \quad \Rightarrow \quad \tilde{u}' = c_1 = \text{const., i.e. OK!}$$

$$\tilde{u} = c_0 + c_2 x^2 \quad \Rightarrow \quad \tilde{u}' = c_2 x \neq \text{const., i.e. NOT OK!}$$

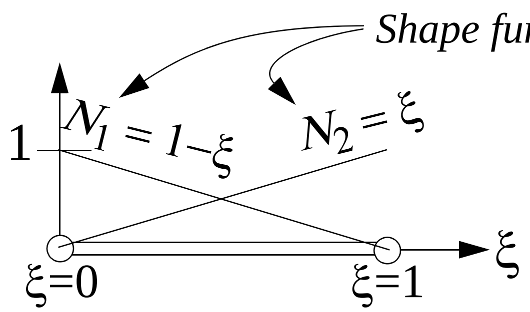
(remedy add the term $c_1 x$)

- ***Approximate solution ansatz function*** $\tilde{u}(x)$***:***

  Formulated by use of ***shape functions***, N_I, and ***node values***, u_I, of the primary variable.

  A ***shape function***, often a ***polynomial***, is expressed as a function of a non-dimensional position coordinate in an element.

E.g. uniaxial problem with linear shape functcion



$$\tilde{u}(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{N}\mathbf{d}_e$$

An approximate solution function based on a polynomial of degree $n-1$, requires n nodes, i.e. one node for each coefficient in the polynomial, giving the interpolation: $\tilde{u}(\xi) = \sum_{I=1}^{n} N_I u_I = \mathbf{N}\mathbf{d}_e$

Properties of shape functions:

Shape fcn. of node I

Coordinate of node J

$$(i) \qquad N_I(\xi_J) = \delta_{IJ} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$

$$(ii) \qquad \sum_{I=1}^{n} N_I = 1$$

(do not apply to problems with rotational d.o.f., e.g. beams)

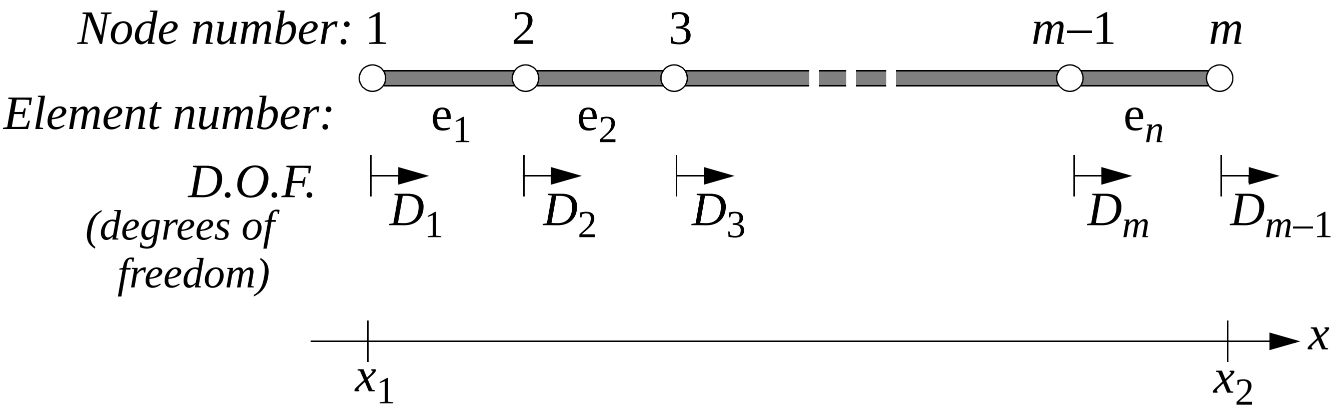
- ***Weight functions*** *v(x)****:***

  Choose piece wise fcn. with the same interpolation as chosen for $\tilde{u}$ (***Galerkin's method***).

  E.g. uniaxial problem with linear shape functcion

$$v(\xi) = N_1(\xi)\beta_1 + N_2(\xi)\beta_2 = = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \mathbf{N}\boldsymbol{\beta}_e$$

Assembly of all *n* elements => FEM-Eq.



Ansatz fcn. & weight fcn.

$$\tilde{u} = \mathbf{N}_G\mathbf{D}$$

$$v = \underbrace{\mathbf{N}_G}\boldsymbol{\beta} = \boldsymbol{\beta}^T\mathbf{N}_G^T$$

Note! contains shape fcn. for all elements!

$$\mathbf{D} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{m-1} \\ D_m \end{bmatrix} \qquad \begin{matrix} e_1 \\ e_2 \\ \\ e_n \end{matrix} \qquad \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{m-1} \\ \beta_m \end{bmatrix} = \boldsymbol{\beta}$$

arbitrary coefficients

Derivatives: $\dfrac{d\tilde{u}}{dx} = \dfrac{d\mathbf{N}_G}{dx}\mathbf{D} = \mathbf{B}_G\mathbf{D}$ $\qquad \dfrac{dv}{dx} = \boldsymbol{\beta}^T\dfrac{d\mathbf{N}_G^T}{dx} = \boldsymbol{\beta}^T\mathbf{B}_G^T$

Inserted into the weak form gives the ***system of FEM-Eqs.***

$$\boldsymbol{\beta}^T\left[\underbrace{\left\{\int_{x_1}^{x_2}\mathbf{B}_G^T E\mathbf{B}_G A\,dx\right\}}_{\mathbf{K}}\mathbf{D} - \underbrace{\left\{[\mathbf{N}_G^T\bar{N}]_{x_1}^{x_2} + \int_{x_1}^{x_2}\mathbf{N}_G^T K_x A\,dx\right\}}_{\mathbf{F}}\right] = 0$$

K ***Stiffness matrix***

F ***External load vector***

Global displacement vector

This can be written as

$$\Rightarrow \begin{bmatrix} \beta_1 & \beta_2 & & \beta_m \end{bmatrix} \begin{bmatrix} \text{Eq. 1} \\ \text{Eq. 2} \\ \\ \\ \text{Eq. } m \end{bmatrix} = 0$$

m equations
for *m* unknowns!

Since all β_i are arbitrary, every single one of the equations must be equal to zero. Thus by the arbitrariness of β_i we obtain

$$[\mathbf{KD} - \mathbf{F}] = 0 \quad \Leftrightarrow \quad \mathbf{KD} = \mathbf{F}$$

In practise, **K** and **F** are evaluated by element wise integration, i.e.

$$K = \int_{x_1^1}^{x_2^1} \mathbf{B}_G^T EA \mathbf{B}_G \mathrm{d}x + \ldots + \int_{x_1^n}^{x_2^n} \mathbf{B}_G^T EA \mathbf{B}_G \mathrm{d}x =$$

$$= \sum_{e=1}^{n} \left[\int_0^1 \mathbf{B}^T EA \mathbf{B} l_e \mathrm{d}\xi \right]_e = \sum_{e=1}^{n} \mathbf{K}_e$$

Volume forces

Point forces acting on nodes enters here!

$$\mathbf{F} = \sum_{e=1}^{n} \left[\int_0^1 \mathbf{N}^T K_x A l_e \mathrm{d}\xi \right]_e + \mathbf{F}_s = \sum_{e=1}^{n} [\mathbf{F}_b]_e + \mathbf{F}_s$$

Summary:

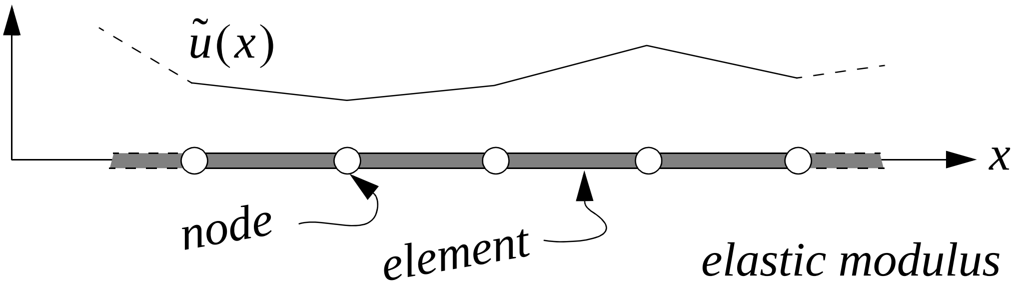
K is obtained by summation of all element stiffness matrices

F is obtained by summation of all distributed loads acting on elements and all forces acting directly on nodes.

This sumation procedure is called the ***assembly procedure***.

Summary: FEM-analysis of trusses (1D)

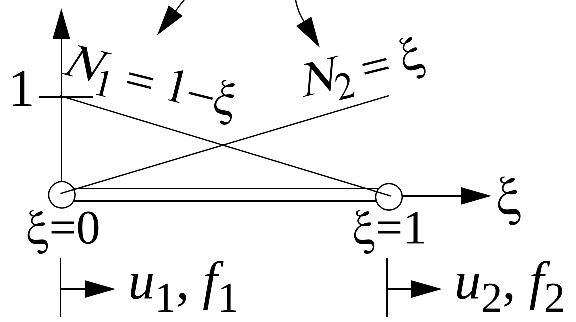
1. Discretization: *divide the truss in elements & nodes and use a simple displacement interpolation in each element!*



area *elastic modulus* *element length*

2. Calculate element matrices, given A, E, l_e (here constants)**:**

shape functions



$$\tilde{u}(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{N}\mathbf{d}_e$$

$$\tilde{\varepsilon}(x) = \frac{d\tilde{u}}{dx} = \frac{d}{dx}\mathbf{N}\mathbf{d}_e = \underbrace{\begin{bmatrix} \frac{-1}{l_e} & \frac{1}{l_e} \end{bmatrix}}_{\mathbf{B}} \mathbf{d}_e$$

$$\mathbf{k}_e = \int_0^1 \mathbf{B}^T E \mathbf{B} A l_e d\xi = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{f}_b = \int_0^1 \mathbf{N}^T K_x A l_e d\xi = \left\{ \begin{matrix} \text{Example:} \\ K_x = q_0 + q_1\xi \end{matrix} \right\} = \frac{Al_e}{2}\begin{bmatrix} q_0 + q_1/3 \\ q_0 + 2q_1/3 \end{bmatrix}$$

3. Assembly: Stiffness matrix & external load vector

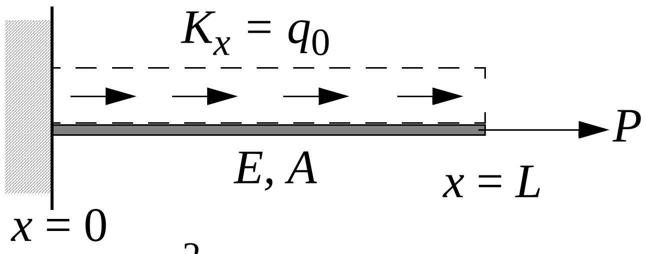
$$\mathbf{K} = \sum_{e=1}^{N_e} \boldsymbol{K}_e \qquad \mathbf{F} = \sum_{e=1}^{N_e} \mathbf{F}_b + \mathbf{F}_s$$

Point forces acting in nodes

4. Introduce B.C. and Solve Eq. System: $\mathbf{KD} = \mathbf{F}$

5. Evaluate the results: (e.g. stresses)

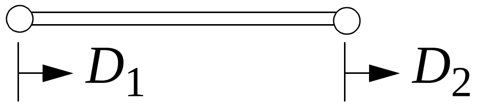
<u>Example:</u>



Exact solution: $$u(x) = \frac{PL}{EA}\cdot\frac{x}{L} + \frac{q_0L^2}{E}\left(\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right)$$

$$\sigma(x) = E\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{P}{A} + q_0L\left(1 - \frac{x}{L}\right)$$

FEM solution (one linear element):



$$\mathbf{K} = \mathbf{k}_e = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Reaction-force

$$\mathbf{F}_b = \frac{ALq_0}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{F}_s = \begin{bmatrix} R \\ P \end{bmatrix}$$

Eq. system (D_1 = 0 => remove row 1 & column 1)

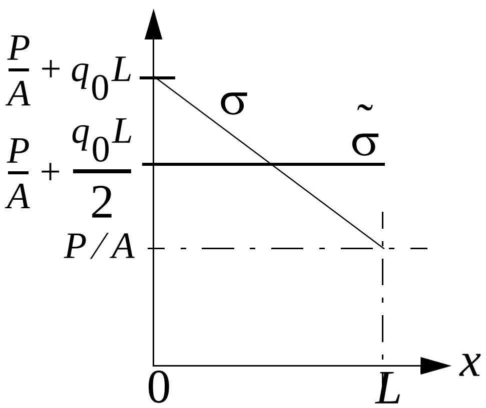
$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} R + ALq_0/2 \\ P + ALq_0/2 \end{bmatrix} \Rightarrow [D_2] = \frac{PL}{EA} + \frac{q_0L^2}{2E}$$

Eq. (1) $$R = \frac{EA}{L}(-D_2) - \frac{ALq_0}{2} = -P - ALq_0$$

Evaluate the result!

$$\tilde{u}(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = 0 + \frac{x}{L}D_2 = \frac{PL}{EA}\frac{x}{L} + \frac{q_0L^2}{2E}\frac{x}{L}$$

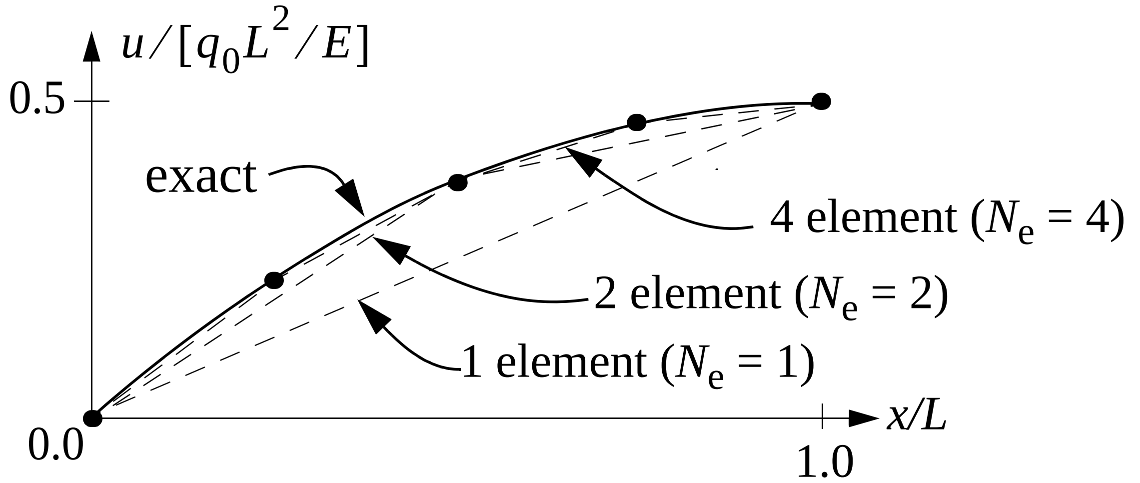
Exact *Approximate*



$$\tilde{\sigma}(x) = E\begin{bmatrix} B_1 & B_2 \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \frac{P}{A} + \frac{q_0L}{2}$$

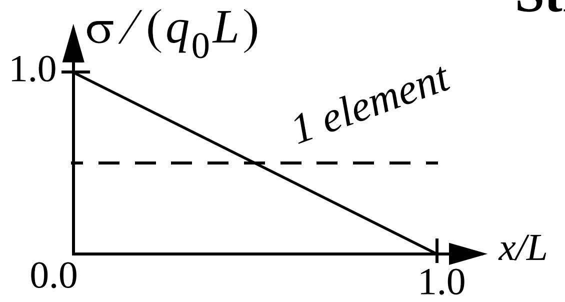
Example cont.:

Displacement

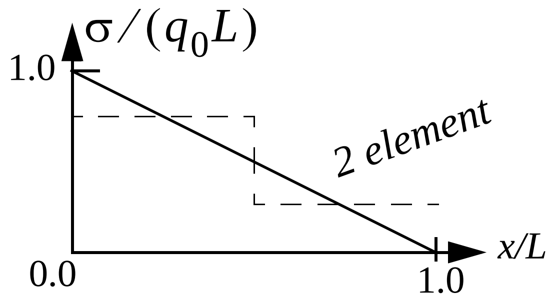


Stress

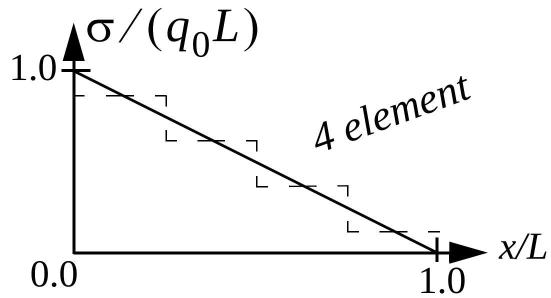
measure of error



$$\left|\frac{\tilde{\sigma}(0)-\sigma(0)}{\sigma(0)}\right| = 50\ \%$$



$$\left|\frac{\tilde{\sigma}(0)-\sigma(0)}{\sigma(0)}\right| = 25\ \%$$



$$\left|\frac{\tilde{\sigma}(0)-\sigma(0)}{\sigma(0)}\right| = 12.5\ \%$$

Element length $\sim 1/N_e$

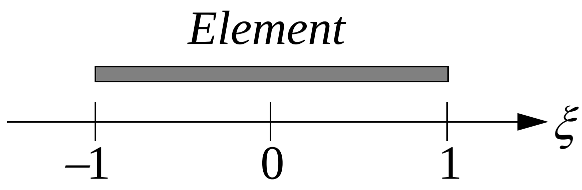
$$Error = \frac{1}{2N_e} \quad \Rightarrow \quad \log(Error) = -\log(2) - \log(N_e)$$

"Linear" **convergence** in stress! — a coincidence?

Higher order truss elements in 1D

(see text book pp. 87–88)

Describe the position in an element by a natural coordinate ξ (non-dimensional)



Approximate displacement interpolation — a polynomial of degree n

$$\tilde{u}(\xi) = a_0 + a_1\xi + \ldots + a_n\xi^n$$

Express using *nodal displacements* d_i & *shape functions* N_i

To determine the $n + 1$ coeff. a_i, $n + 1$ nodes are needed

$$\Rightarrow \tilde{u}(\xi) = N_1(\xi)d_1 + \ldots + N_{n+1}(\xi)d_{n+1} = \mathbf{N}\mathbf{d}_e$$

Features of shape functions: *(i)* $N_i(\xi_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

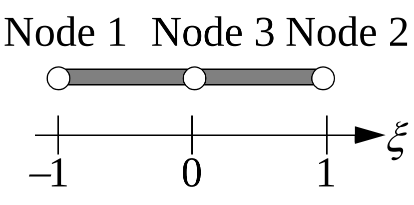
(see text book pp. 41–52)

(ii) $N_1 + \ldots + N_{n+1} = 1$

Lagrange interpolation satisfy these requirements, i.e. the shape fcn. at node k (position $\xi = \xi_k$) can be determined as: $N_k = l_k^n(\xi)$

$$l_k^n(\xi) = \prod_{\substack{i=1 \\ i \neq k}}^{i=n+1} \frac{(\xi - \xi_i)}{(\xi_k - \xi_i)} = \frac{(\xi - \xi_1)\ldots(\xi - \xi_{k-1})(\xi - \xi_{k+1})\ldots(\xi - \xi_{n+1})}{(\xi_k - \xi_1)\ldots(\xi_k - \xi_{k-1})(\xi_k - \xi_{k+1})\ldots(\xi_k - \xi_{n+1})}$$

Ex.: quadratic element *(n* = 2), with nodal points at: $\xi_k = \{-1, 0, 1\}$



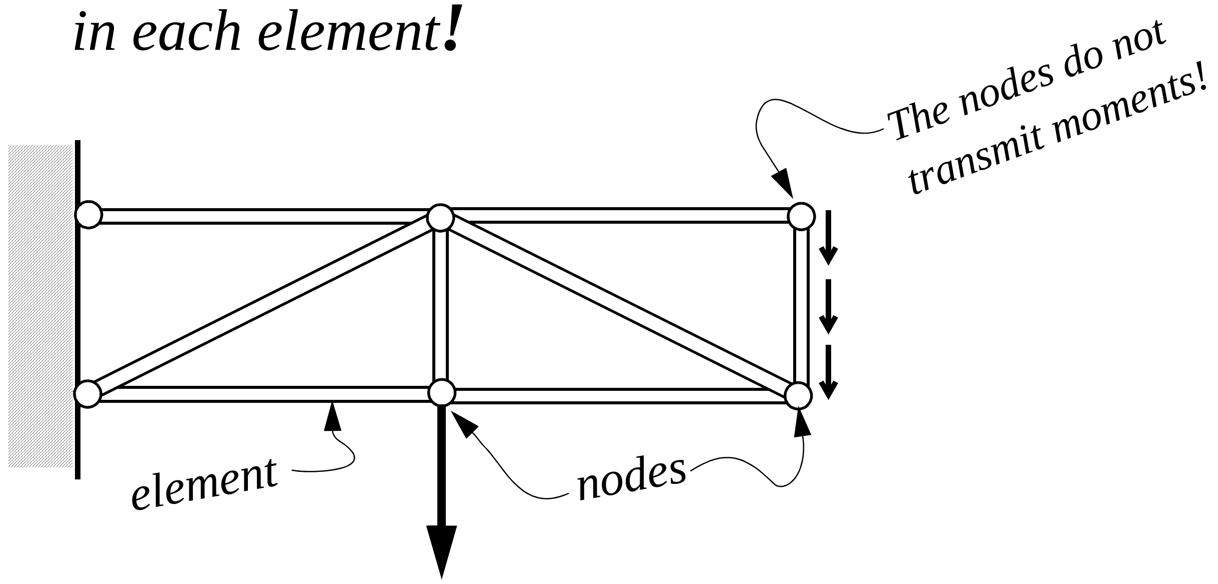
$$N_1 = l_{k=1}^{n=2}(\xi) = \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} = \frac{\xi(\xi - 1)}{2}$$

$$N_2 = l_{k=2}^{n=2}(\xi) = \frac{(\xi - (-1))(\xi - 0)}{(1 - (-1))(1 - 0)} = \frac{\xi(\xi + 1)}{2}$$

$$N_3 = l_{k=3}^{n=2}(\xi) = \frac{(\xi - (-1))(\xi - 1)}{(0 - (-1))(0 - 1)} = 1 - \xi^2$$

Procedure for FEM-analysis of truss structures

1. Discretization: *divide the truss structure into elements & nodes and use a simple displacement interpolation in each element!*

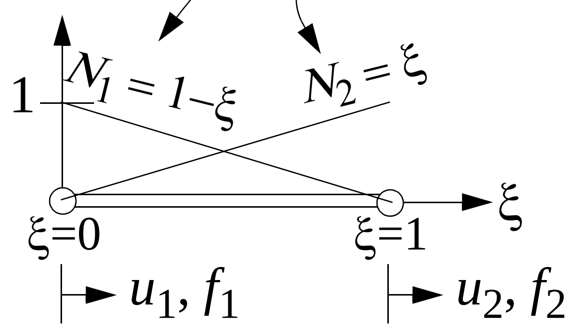


elastic modulus

area *element length*

2. Calculate element matrices, given A, E, l_e (here constants)**:**

shape functions



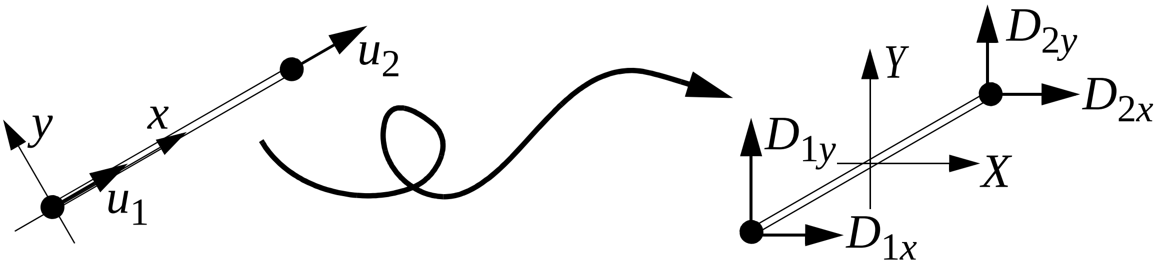
$$\tilde{u}(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{N}\mathbf{d}_e$$

$$\tilde{\varepsilon}(x) = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}\mathbf{d}_e = \underbrace{\begin{bmatrix} \frac{-1}{l_e} & \frac{1}{l_e} \end{bmatrix}}_{\mathbf{B}} \mathbf{d}_e$$

$$\mathbf{k}_e = \int_0^1 \mathbf{B}^T E \mathbf{B} A l_e \mathrm{d}\xi = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{f}_b = \int_0^1 \mathbf{N}^T K_x A l_e \mathrm{d}\xi = \left\{ \begin{matrix} \text{Example:} \\ K_x = q_0 + q_1\xi \end{matrix} \right\} = \frac{A l_e}{2} \begin{bmatrix} q_0 + q_1/3 \\ q_0 + 2q_1/3 \end{bmatrix}$$

3. Transformation: local $\{x,y\}$ — global coord. syst. $\{X,Y\}$



$$\mathbf{d}_e = \mathbf{T}\mathbf{D}_e \qquad \mathbf{T} = \begin{bmatrix} l_{12} & m_{12} & 0 & 0 \\ 0 & 0 & l_{12} & m_{12} \end{bmatrix}$$

$$\mathbf{F}_b = \mathbf{T}^T\mathbf{f}_b$$

$$\mathbf{K}_e = \mathbf{T}^T\mathbf{k}_e\mathbf{T} = \frac{EA}{l_e}\begin{bmatrix} l_{12}^2 & l_{12}m_{12} & -l_{12}^2 & -l_{12}m_{12} \\ l_{12}m_{12} & m_{12}^2 & -l_{12}m_{12} & -m_{12}^2 \\ -l_{12}^2 & -l_{12}m_{12} & l_{12}^2 & l_{12}m_{12} \\ -l_{12}m_{12} & -m_{12}^2 & l_{12}m_{12} & m_{12}^2 \end{bmatrix}$$

4. Assembly: Stiffness matrix & load vector

$$\mathbf{K} = \sum_{e=1}^{N_e} \boldsymbol{K}_e \qquad \mathbf{F} = \sum_{e=1}^{N_e} \mathbf{F}_b + \mathbf{F}_s$$

Point forces acting in nodes

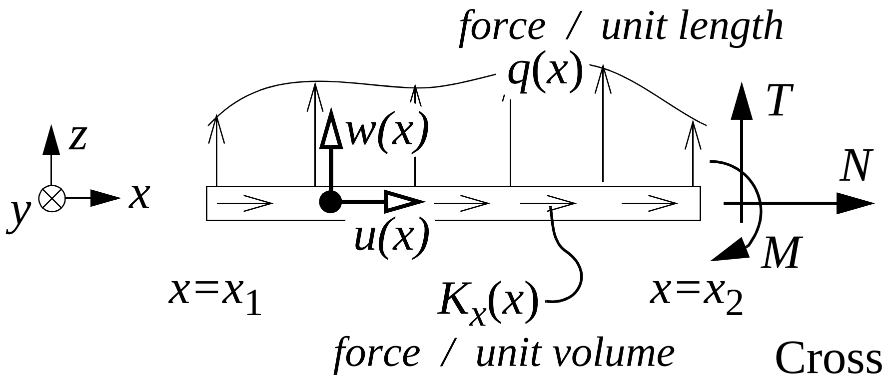
5. Introduce B.C. and Solve Eq. System: $\mathbf{K}\mathbf{D} = \mathbf{F}$

6. Evaluate the results: (e.g. stresses)

$$\tilde{\sigma}(x) = E\tilde{\varepsilon}(x) = E\mathbf{B}\mathbf{d}_e = E\frac{(u_2 - u_1)}{l_e}$$

Lecture 8 & 9

FEM-Eq. for a Beam



Cross section: *I(x), A(x)*

Material: *E(x)*

Strong form (local form):

<u>Equilibrium:</u>

$$T' = -q$$
$$M' = T$$
$$N' = -K_x A$$

<u>Constitutive relations:</u>

$$M = -EIw''$$
$$N = EAu'$$

strain

$\Rightarrow$

"Truss Eq."

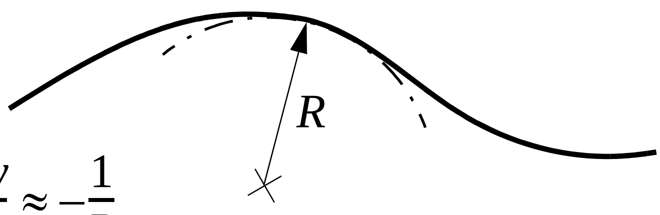
$$(EAu')' + AK_x = 0$$

"Beam Eq."

<u>O.D.E.:</u> $(EIw'')'' - q = 0$

<u>Boundary Conditions:</u>

$$w = \bar{w} \quad \text{or} \quad (EIw'')' = -\bar{T}$$
$$w' = \bar{w}' \quad \text{or} \quad EIw'' = -\bar{M}$$



curvature: $w'' = \dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2} \approx -\dfrac{1}{R}$

<u>Requirements on the solution:</u> the deflection, *w(x)*, and its derivative, d*w*/d*x*, must be continuos functions

Weak form (variational form, integral form):

1. Multiply **O.D.E.** and **B.C.** by an ***arbitrary weight function***, *v(x)*, and integrate over the length of the beam:

$$\Rightarrow \begin{cases} \int_{x_1}^{x_2} v(x)[(EIw'')'' - q]\mathrm{d}x = 0 & \text{(1a)} \\ [v'(x)(\overline{M} + EIw'')]\big|_{x_{\mathrm{N.R.V}}} = 0 & \text{(1b)} \\ [v(x)(\overline{T} + (EIw'')')]\big|_{x_{\mathrm{N.R.V}}} = 0 & \text{(1c)} \\ \text{Choose } v = 0 \text{ on boundaries with essential B.C.} & \text{(1d)} \end{cases}$$

2. Integrate the first term in (1a) by parts twice, i.e. lower $w^{iv} \to w''$

use $\quad [vf]_{x_1}^{x_2} = \int_{x_1}^{x_2} [vf]'\mathrm{d}x = \int_{x_1}^{x_2} v'f\mathrm{d}x + \int_{x_1}^{x_2} vf'\mathrm{d}x$

$$\int_{x_1}^{x_2} v(EIw'')''\mathrm{d}x = [v(EIw'')']_{x_1}^{x_2} - \int_{x_1}^{x_2} v'(EIw'')'\mathrm{d}x$$

$$= [v(EIw'')']_{x_1}^{x_2} - \left\{ [v'(EIw'')]_{x_1}^{x_2} - \int_{x_1}^{x_2} v''(EIw'')\mathrm{d}x \right\}$$

inserted into (1a) gives

$$\int_{x_1}^{x_2} v''EIw''\mathrm{d}x + \underbrace{[v(EIw'')']_{x_1}^{x_2}}_{v(-\overline{T})\ \text{Eq.(1c,d)}} - \underbrace{[v'(EIw'')]_{x_1}^{x_2}}_{v'(-\overline{M})\ \text{Eq.(1b,d)}} - \int_{x_1}^{x_2} vq\mathrm{d}x = 0$$

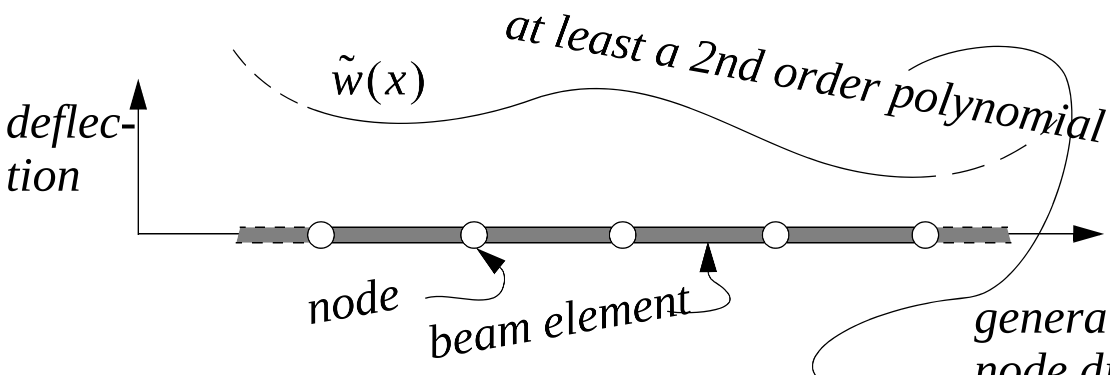
Weak form, definition: Find *w(x)* among admissible functions that satisfy essential B.C. ($w = \overline{w}$, $w' = \overline{w}'$), such that

$$\int_{x_1}^{x_2} v''EIw''\mathrm{d}x - \left[[v\overline{T}]_{x_1}^{x_2} - [v'\overline{M}]_{x_1}^{x_2} + \int_{x_1}^{x_2} vq\mathrm{d}x \right] = 0$$

for an arbitrary $v(x)$, with $v = 0$ on boundaries with essential B.C.

Thus, the approximate displacement interpolation (deflection), *w(x)*, and the weight fcn., *v(x)*, must be twice differentiable!

FEM-Eq.—divide into elements (discretization):



generalized node displace- ment vector

"shape functions"

Displ. interpolation in each element: $\tilde{w}(x) = \mathbf{N}(x)\mathbf{d}_e$

Weight function (Galerkin): $v(x) = \mathbf{N}(x)\boldsymbol{\beta} = \boldsymbol{\beta}^T\mathbf{N}^T(x)$

arbitrary vector

2nd order derivatives:

$$\tilde{w}''(x) = \mathbf{N}''\mathbf{d}_e = \mathbf{B}\mathbf{d}_e$$

$$v''(x) = \boldsymbol{\beta}^T(\mathbf{N}'')^T = \boldsymbol{\beta}^T\mathbf{B}^T$$

Insert *w(x)* and *v(x)* into Eq. (3):

$$\boldsymbol{\beta}^T\left[\int_{x_1}^{x_2}\mathbf{B}^T EI\mathbf{B}\,\mathrm{d}x\right]\mathbf{d}_e = \boldsymbol{\beta}^T\left[[\mathbf{N}^T T]_{x_1}^{x_2} - [(\mathbf{N}')^T M]_{x_1}^{x_2} + \int_{x_1}^{x_2}\mathbf{N}^T q\,\mathrm{d}x\right]$$

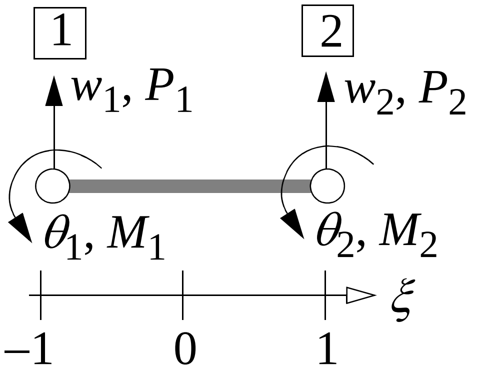
Shorten with vector $\boldsymbol{\beta}^T$ => **FEM-Eq. for a beam element**

$$\underbrace{\left[\int_{x_1}^{x_2}\mathbf{B}^T EI\mathbf{B}\,\mathrm{d}x\right]}_{\substack{\textit{Stiffness matrix}\\ n\times n}}\mathbf{d}_e = \underbrace{\left[[\mathbf{N}^T T]_{x_1}^{x_2} - [(\mathbf{N}')^T M]_{x_1}^{x_2} + \int_{x_1}^{x_2}\mathbf{N}^T q\,\mathrm{d}x\right]}_{\substack{\textit{Load vector}\\ n\times 1}}$$

Displacement vector
$n \times 1$

2-node beam element

(3rd order polynomial for the deflection)



Element length: $2a$

Moment of inertia: I

Elastic modulus: E

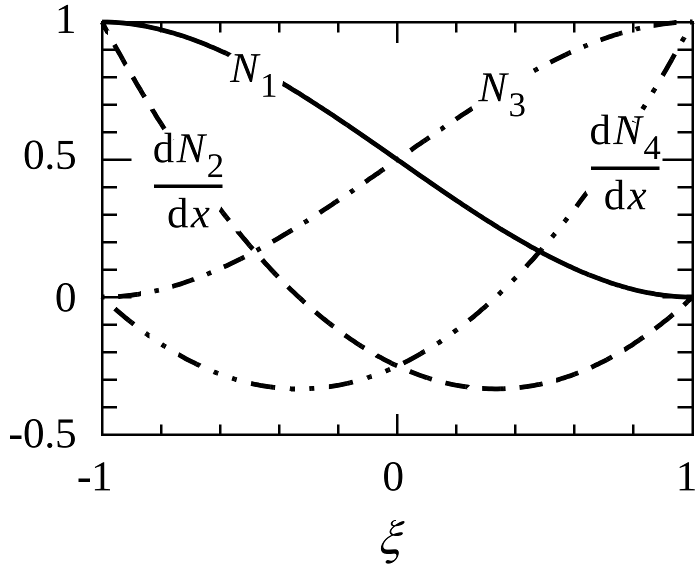
$$\tilde{w}(\xi) = \underbrace{\begin{bmatrix} N_1(\xi) & N_2(\xi) & N_3(\xi) & N_4(\xi) \end{bmatrix}}_{\mathbf{N}(\xi)} \underbrace{\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}}_{\mathbf{d}_e} \qquad \tilde{\theta}(\xi) = \frac{1}{a}\frac{\mathrm{d}\mathbf{N}}{d\xi}\mathbf{d}_e$$

Shape functions:

$$N_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \qquad N_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)a$$

$$N_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \qquad N_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)a$$

Note that! $N_i(\xi_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ for $i, j = 1, 3$ $\qquad N'_i(\xi_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ for $i, j = 2, 4$



Shape fcn. inserted into the weak form gives the FEM Eq.

$$\underbrace{\left[\frac{EI}{a^3}\int_{-1}^{1}(\mathbf{N}'')^T\mathbf{N}''\mathrm{d}\xi\right]}_{\mathbf{k}_e}\mathbf{d}_e = \underbrace{\left[\underbrace{\begin{bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{bmatrix}}_{\mathbf{f}_s} + \underbrace{a\int_{-1}^{1}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q(\xi)\mathrm{d}\xi}_{\mathbf{f}_b}\right]}_{\mathbf{f}_e}$$

$$\mathbf{f}_e = \mathbf{f}_s + \mathbf{f}_b$$

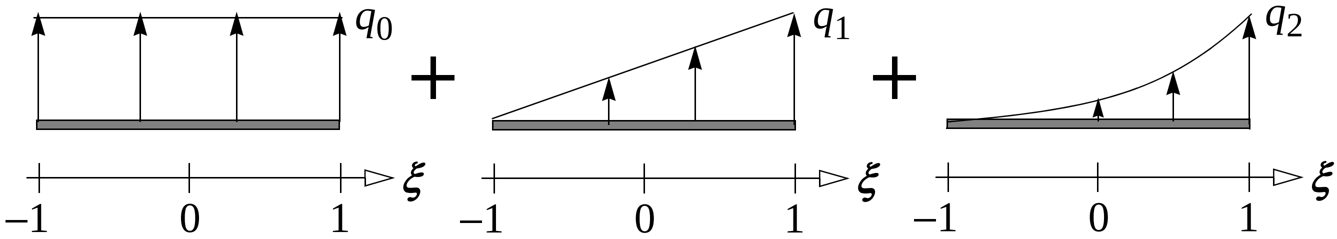
Element stiffness matrix:

$$\mathbf{k}_e = \frac{EI}{a^3}\int_{-1}^{1}(\mathbf{N}'')^T\mathbf{N}''\mathrm{d}\xi = \frac{EI}{2a^3}\begin{bmatrix} 3 & 3a & -3 & 3a \\ 3a & 4a^2 & -3a & 2a^2 \\ -3 & -3a & 3 & -3a \\ 3a & 2a^2 & -3a & 4a^2 \end{bmatrix}$$

Element nodal force vector, contribution from distributed load:

$$\mathbf{f}_b = a\int_{-1}^{1}\mathbf{N}^T q(\xi)\mathrm{d}\xi = a\int_{-1}^{1}\begin{bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \\ N_4(\xi) \end{bmatrix} q(\xi)\mathrm{d}\xi$$

Example:



$$\Rightarrow q(\xi) = q_0 + q_1\left(\frac{1+\xi}{2}\right) + q_2\left(\frac{1+\xi}{2}\right)^2$$

Nodal force vector: $\mathbf{f}_b = aq_0\begin{bmatrix} 1 \\ a/3 \\ 1 \\ -a/3 \end{bmatrix} + \frac{aq_1}{30}\begin{bmatrix} 9 \\ 4a \\ 21 \\ -6a \end{bmatrix} + \frac{aq_2}{15}\begin{bmatrix} 2 \\ a \\ 8 \\ -2a \end{bmatrix}$

force

moment

Repetition

FEM-analysis: Computational steps

1. Spatial discretization: introduce nodes (D.O.F.) and divide the structure into elements (*pre-processing*)

2. (a) Calculate the element stiffness matrix, $\mathbf{k}_e$, and the element load vector, $\mathbf{f}_b$, for each element

(b) Coordinate transformation: local–global

$$\mathbf{K}_e = \mathbf{T}^T \mathbf{k}_e \mathbf{T}$$

$$\mathbf{F}_b = \mathbf{T}^T \mathbf{f}_b$$

3. Assembly of all element (total number = N_e) stiffness matrices & load vectors

$$\mathbf{K} = \sum_{e=1}^{N_e} \mathbf{K}_e \qquad \mathbf{F} = \sum_{e=1}^{N_e} \mathbf{F}_b + \mathbf{F}_s$$

Point forces acting in nodes

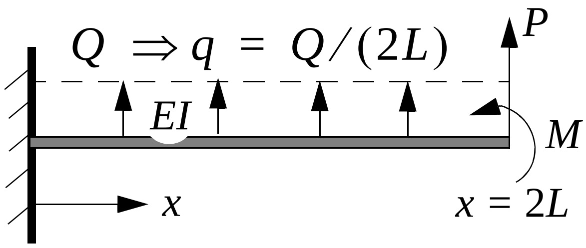
4. Introduce boundary conditions & solve equation system:

$$\mathbf{KD} = \mathbf{F}$$

5. Evaluate the result (*post-processing*)

* Reaction forces, cross section quantities, etc.
* Stresses

Example: Cantilever beam

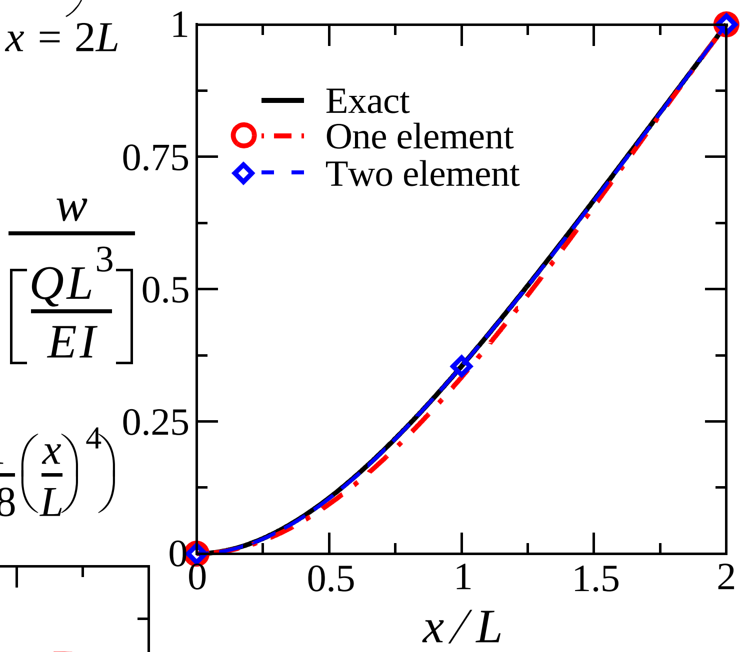


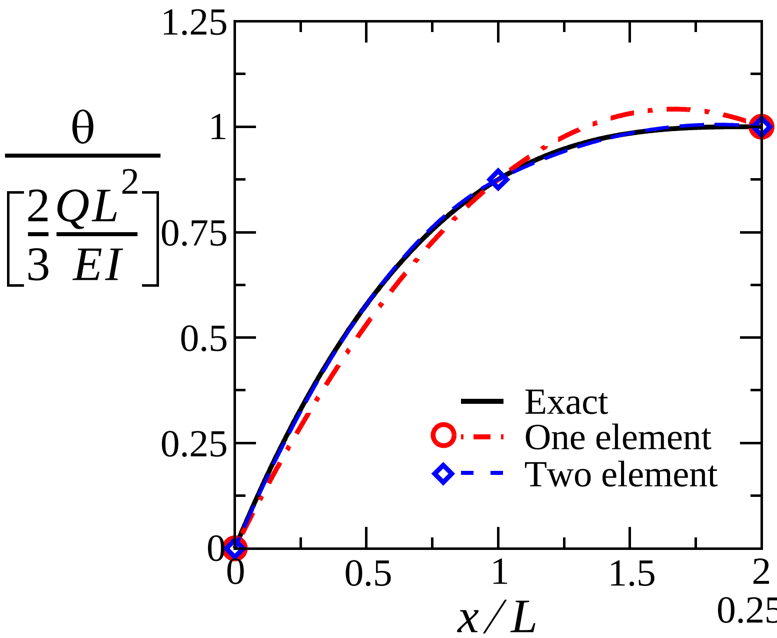
Deflection (exact solution):

$$w(x) = \frac{PL^3}{6EI}\left(6\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right)$$

$$+ \frac{ML^2}{2EI}\left(\frac{x}{L}\right)^2$$

$$+ \frac{QL^3}{EI}\left(\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{48}\left(\frac{x}{L}\right)^4\right)$$



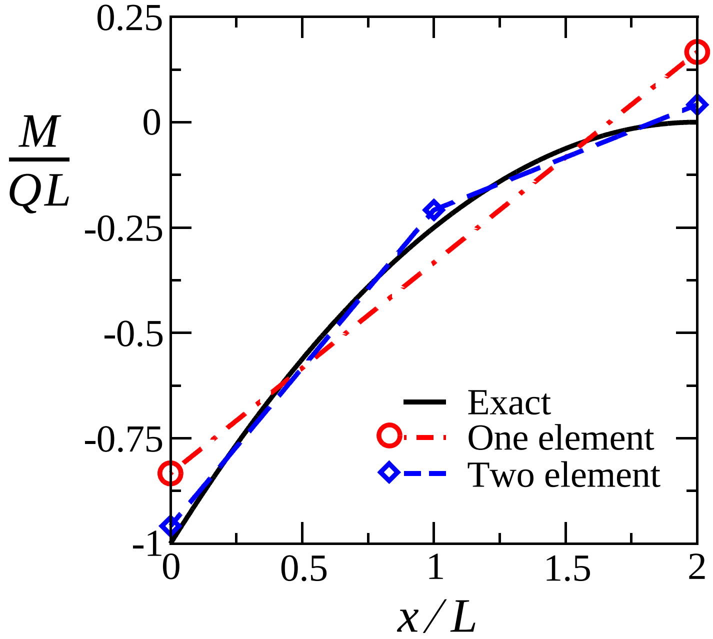


Angle (slope):

$$\theta(x) = \frac{\mathrm{d}w}{\mathrm{d}x}$$

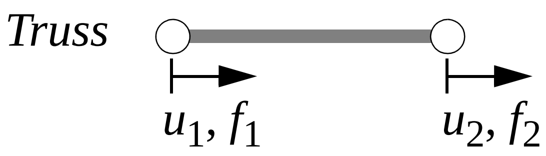
Bending moment:

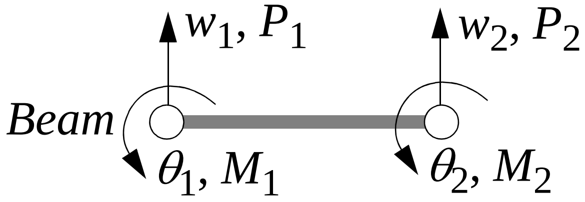
$$M(x) = -EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}$$



Truss/Beam problem: Features of FEM-solutions

Consider FEM-solutions based on 2-node elements:





$$\mathbf{d}_e^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \qquad \mathbf{d}_e^T = \begin{bmatrix} w_1 & \theta_1 & w_2 & \theta_2 \end{bmatrix}$$

For cases with constant tensile/bending stiffness (truss: *EA* = *const.*; beam: *EI* = *const.*) the nodal displacement vector will be identical with the exact solution.

Reason:

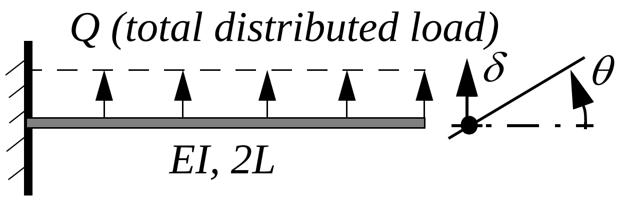
1. The approximate displacement interpolation satisfy the homogeneous solution of the differential equations of the problem

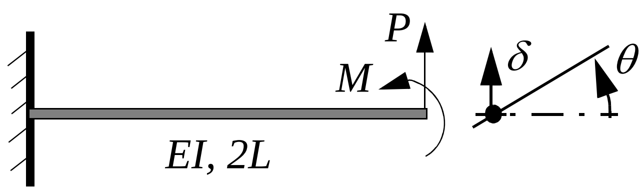
$$\textit{Truss:}\ (EAu')' = 0 \quad \Rightarrow \quad u(x) = c_0 + c_1 x$$

$$\textit{Beam:}\ (EIw'')'' = 0 \Rightarrow w(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

2. A distributed load is replaced by consistent nodal forces, which gives an equivalent problem regarding nodal displacements

"Equivalent Problem"





$$\delta = \frac{QL^3}{EI}$$

$$\theta = \frac{2}{3}\frac{QL^2}{EI}$$

consistent nodal forces

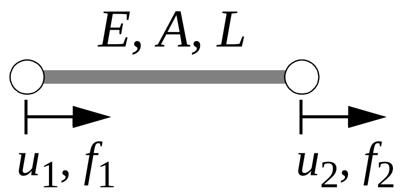
$$P = \frac{Q}{2}; \quad M = -\frac{QL}{6}$$

$$\Rightarrow \delta = \frac{8}{3}\frac{PL^3}{EI} + 2\frac{ML^2}{EI} = \frac{QL^3}{EI}$$

$$\Rightarrow \theta = 2\frac{PL^2}{EI} + 2\frac{ML}{EI} = \frac{2}{3}\frac{QL^2}{EI}$$

Evaluation of normal stress, σ

Truss (bar)



$$\mathbf{d}_e^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

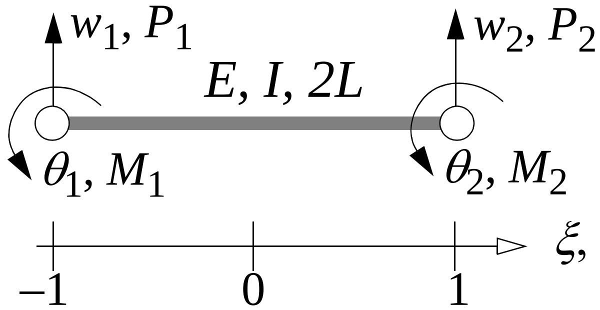
Shape fcn. for truss (1×2)

$$\sigma = E\varepsilon = E\frac{\mathrm{d}u}{\mathrm{d}x} = E\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{N}\mathbf{d}_e) = E\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\mathbf{d}_e = E\mathbf{B}\mathbf{d}_e$$

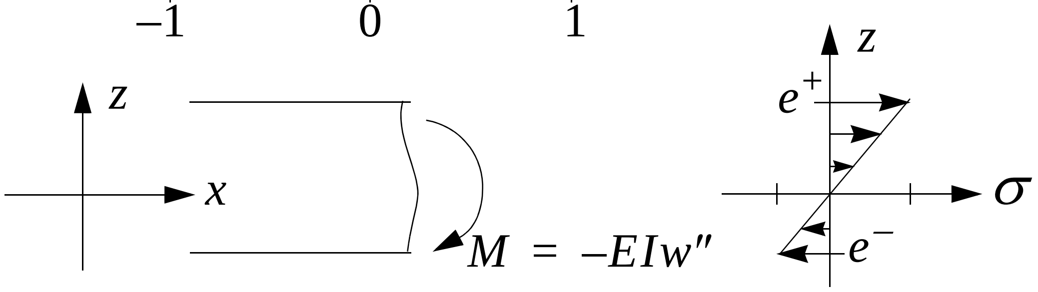
where $\mathbf{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}$

$$\Rightarrow \sigma = E(B_1 u_1 + B_2 u_2) = E\left(\frac{u_2 - u_1}{L}\right)$$

Beam



$$\mathbf{d}_e^T = \begin{bmatrix} w_1 & \theta_1 & w_2 & \theta_2 \end{bmatrix}$$



$\max\{e^+, |e^-|\}$

Shape fcn. for beam (1×4)

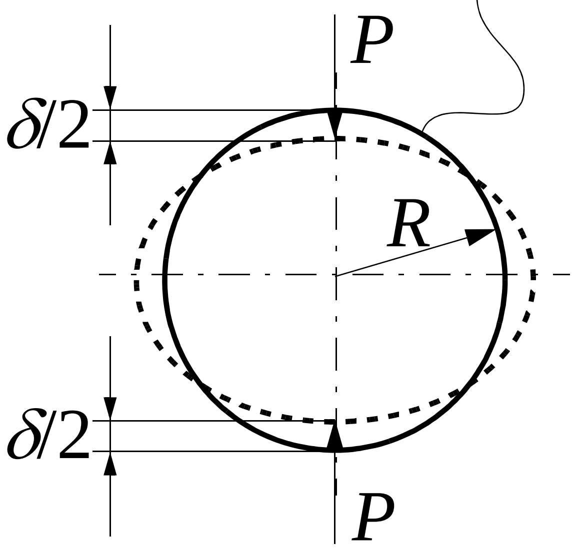
$$\sigma_{\max} = \frac{|M|}{I}e_{\max} = Ee_{\max}|w''| = Ee_{\max}\left|\frac{\mathrm{d}^2\mathbf{N}}{\mathrm{d}x^2}\mathbf{d}_e\right| = Ee_{\max}|\mathbf{B}\mathbf{d}_e|$$

where $\mathbf{B} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} = \begin{bmatrix} \dfrac{3\xi}{2L^2} & \dfrac{(3\xi - 1)}{2L} & \dfrac{-3\xi}{2L^2} & \dfrac{(3\xi + 1)}{2L} \end{bmatrix}$

$$\Rightarrow \sigma_{\max} = Ee_{\max}|B_1 w_1 + B_2\theta_1 + B_3 w_2 + B_4\theta_2|$$

Lecture 10
In-plane frames (beam structures in 2D)

An introductory example ...



"Flexible ring"

Stiffness: $k = P/\delta$ **?**

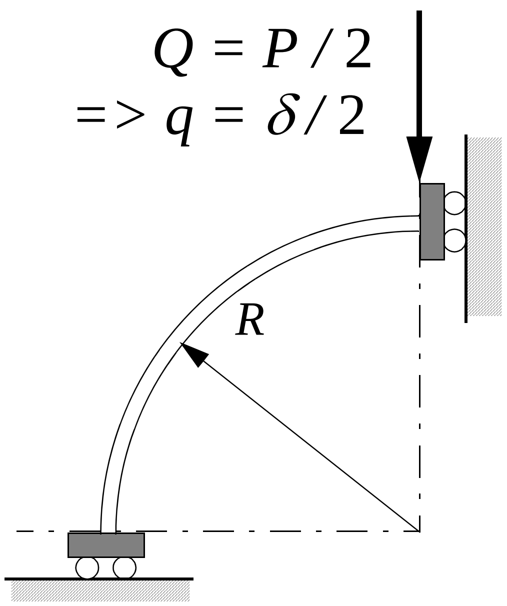
Modelling**?**

Analysis method**?**

Modelling: utilize symmetry!

=> enough to model 1/4

$Q = P/2$
$\Rightarrow q = \delta/2$



Possible ***analysis*** methods:

1. Energy method, $\overline{W}(Q)$

$$\Rightarrow k = \frac{P}{\delta} = \frac{Q}{q} = \frac{Q}{\partial\overline{W}/(\partial Q)}$$

2. FEM, planar frame (2*D*)

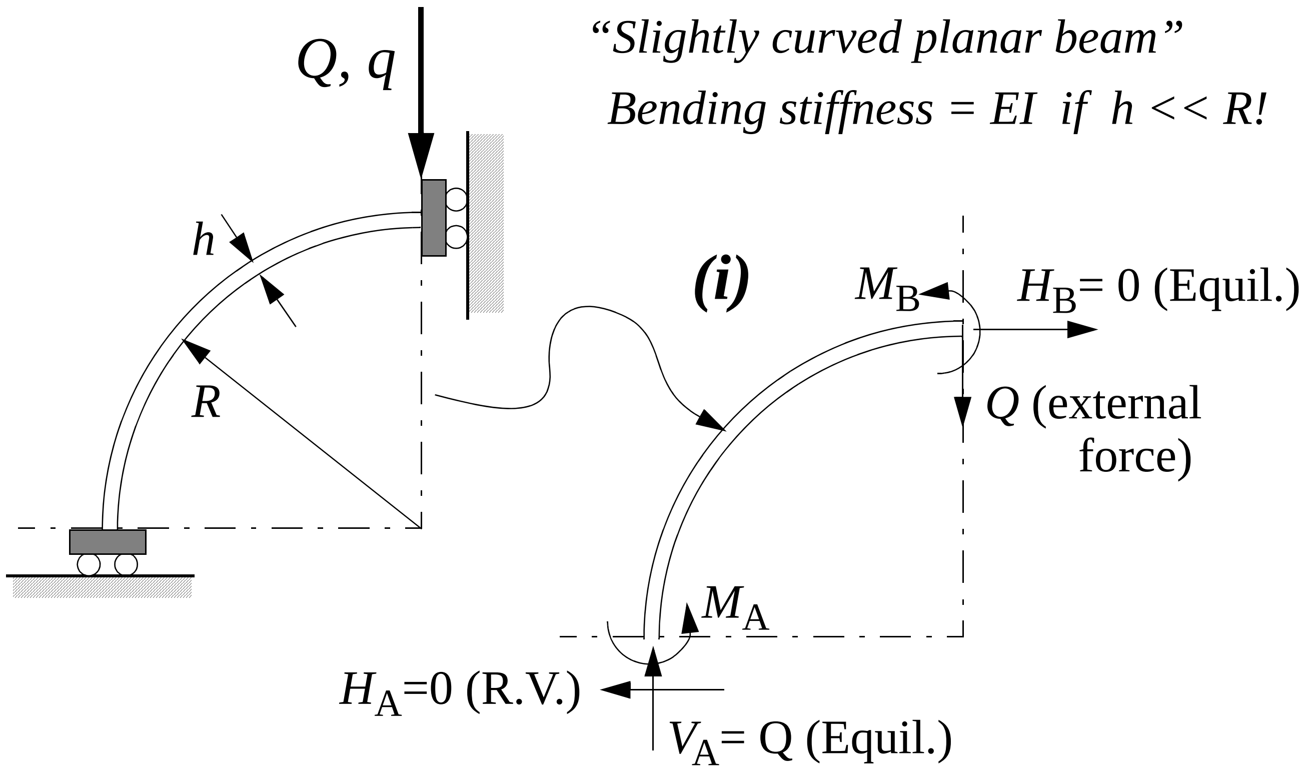
1. Energy method:

(i) free body diagram (introduce reaction forces & identify static indeterminate quantities)

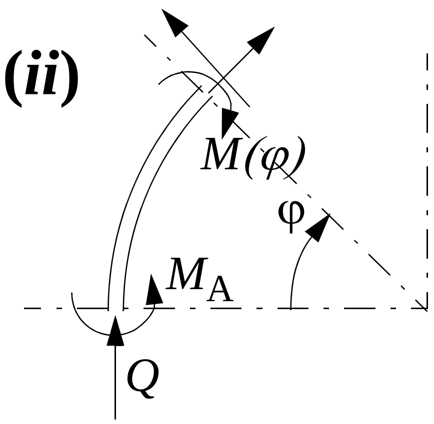
(ii) cut (determine the moment in the beam)

"Slightly curved planar beam"

Bending stiffness = EI if h << R!



Equil.: $M_A + M_B - QL = 0$ *=> 1 static indeterminate!*



Equil.: $M(\varphi) = M_A - QR(1-\cos(\varphi))$

Complementary elastic energy:

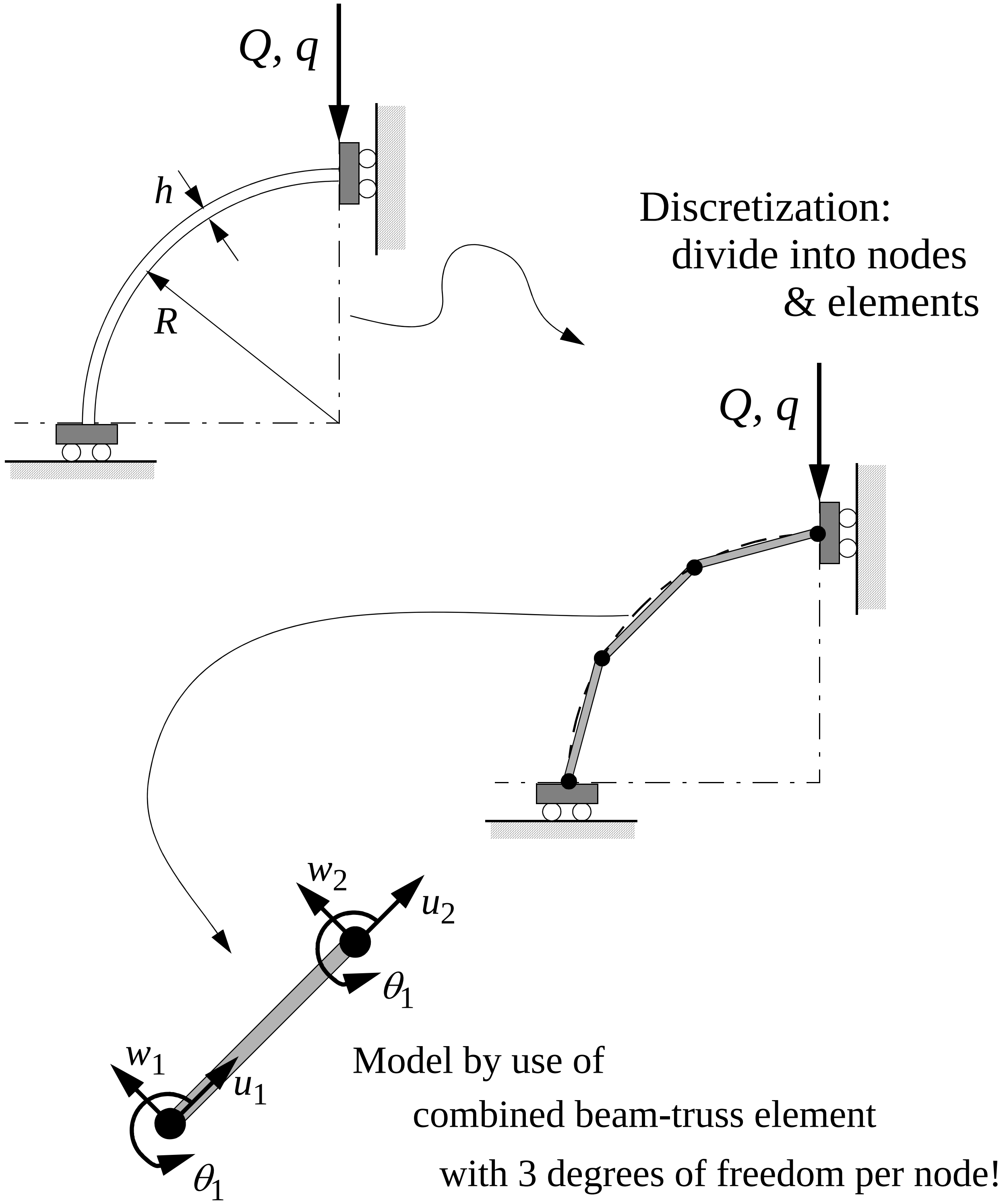
$$\overline{W} = \int_0^{\pi/2} \frac{M(\varphi)^2}{2EI} R \, d\varphi$$

Castigliano's 2:a theorem:

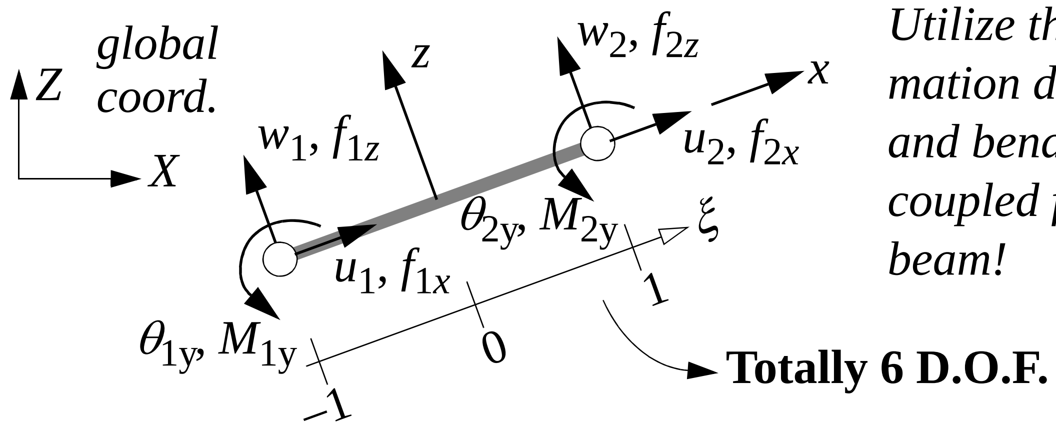
$$\frac{\partial \overline{W}}{\partial M_A} = 0 \Rightarrow M_A = QR\frac{(\pi-2)}{\pi}$$

$$q = \frac{\partial \overline{W}}{\partial Q} = \frac{QR^3}{EI}\frac{(\pi^2-8)}{8\pi}$$

2. FEM:


Q, q
h
R
Discretization:
divide into nodes
& elements
Q, q
w2
u2
θ1
w1
u1
θ1
Model by use of
combined beam-truss element
with 3 degrees of freedom per node!

Combined beam-truss element



Utilize that the deformation due to tension and bending is uncoupled for a straight beam!

Deformation in bending:

$$w(\xi) = \begin{bmatrix} 0 & N_1^b(\xi) & N_2^b(\xi) & 0 & N_3^b(\xi) & N_4^b(\xi) \end{bmatrix} \begin{bmatrix} u_1 \\ w_1 \\ \theta_{1y} \\ u_2 \\ w_2 \\ \theta_{2y} \end{bmatrix}$$

$$\mathbf{k}_e^b = \frac{EI}{2a^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3a & 0 & -3 & 3a \\ 0 & 3a & 4a^2 & 0 & -3a & 2a^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3a & 0 & 3 & -3a \\ 0 & 3a & 2a^2 & 0 & -3a & 4a^2 \end{bmatrix} \qquad \mathbf{f}_e^b = \begin{bmatrix} 0 \\ f_{1z} \\ M_{1y} \\ 0 \\ f_{2z} \\ M_{2y} \end{bmatrix} + a\int_{-1}^{1} \begin{bmatrix} 0 \\ N_1^b \\ N_2^b \\ 0 \\ N_3^b \\ N_4^b \end{bmatrix} q(\xi)\mathrm{d}\xi$$

Deformation in tension:

$$u(\xi) = \begin{bmatrix} \underset{(1-\xi)/2}{N_1^d(\xi)} & 0 & 0 & \underset{(1+\xi)/2}{N_2^d(\xi)} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ w_1 \\ \theta_{1y} \\ u_2 \\ w_2 \\ \theta_{2y} \end{bmatrix}$$

$$\mathbf{k}_e^d = \frac{EA}{2a}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{f}_e^d = \begin{bmatrix} f_{1x} \\ 0 \\ 0 \\ f_{2x} \\ 0 \\ 0 \end{bmatrix} + \int_{-1}^{1} \begin{bmatrix} N_1^d \\ 0 \\ 0 \\ N_2^d \\ 0 \\ 0 \end{bmatrix} K_x(\xi)A(2a)\mathrm{d}\xi$$

Total stiffness in the local coordinate system:

Element stiffness matrix:

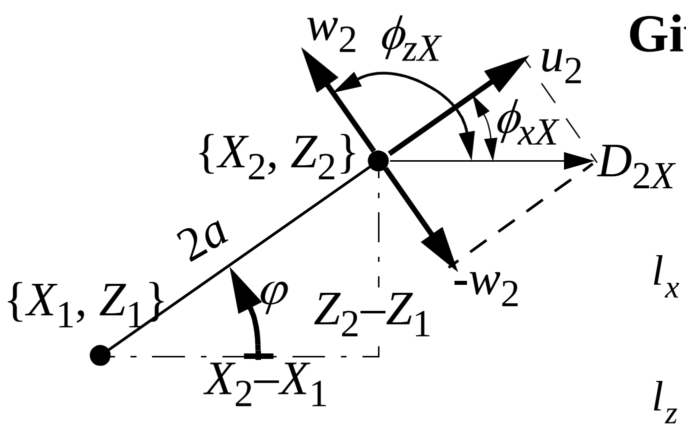
$$\mathbf{k}_e = \mathbf{k}_e^d + \mathbf{k}_e^b =$$

$$= \begin{bmatrix} \frac{EA}{2a} & 0 & 0 & -\frac{EA}{2a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{2a} & 0 & 0 & \frac{EA}{2a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3EI}{2a^3} & \frac{3EI}{2a^2} & 0 & -\frac{3EI}{2a^3} & \frac{3EI}{2a^2} \\ 0 & \frac{3EI}{2a^2} & \frac{2EI}{a} & 0 & -\frac{3EI}{2a^2} & \frac{EI}{a} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3EI}{2a^3} & -\frac{3EI}{2a^2} & 0 & \frac{3EI}{2a^3} & -\frac{3EI}{2a^2} \\ 0 & \frac{3EI}{2a^2} & \frac{EI}{a} & 0 & -\frac{3EI}{2a^2} & \frac{2EI}{a} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{EA}{2a} & 0 & 0 & -\frac{EA}{2a} & 0 & 0 \\ 0 & \frac{3EI}{2a^3} & \frac{3EI}{2a^2} & 0 & -\frac{3EI}{2a^3} & \frac{3EI}{2a^2} \\ 0 & \frac{3EI}{2a^2} & \frac{2EI}{a} & 0 & -\frac{3EI}{2a^2} & \frac{EI}{a} \\ -\frac{EA}{2a} & 0 & 0 & \frac{EA}{2a} & 0 & 0 \\ 0 & -\frac{3EI}{2a^3} & -\frac{3EI}{2a^2} & 0 & \frac{3EI}{2a^3} & -\frac{3EI}{2a^2} \\ 0 & \frac{3EI}{2a^2} & \frac{EI}{a} & 0 & -\frac{3EI}{2a^2} & \frac{2EI}{a} \end{bmatrix}$$

Element load vector: $\mathbf{f}_e = \mathbf{f}_e^d + \mathbf{f}_e^b$

Transformation: local –> global coordinate system (2*D*)

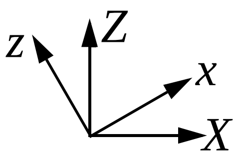


Given D_{2X}, determine u_2 and w_2:

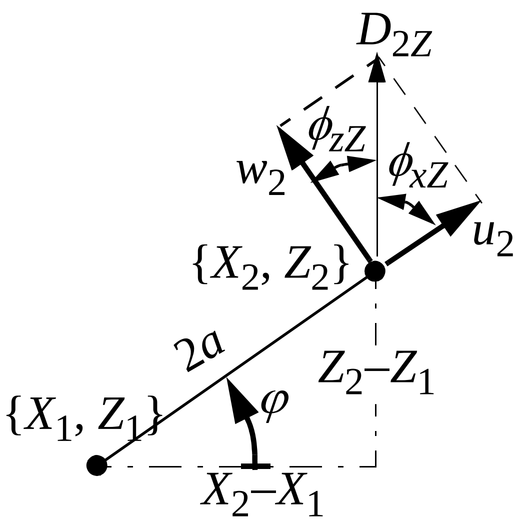
$$l_x = \cos\phi_{xX} = \frac{X_2 - X_1}{2a} = \cos\varphi$$

$$l_z = \cos\phi_{zX} = -\frac{Z_2 - Z_1}{2a} = -\sin\varphi$$

$$\Rightarrow u_2 = D_{2x} l_x, \quad w_2 = D_{2x} l_z$$



Given D_{2Z}, determine u_2 and w_2:



$$m_x = \cos\phi_{xZ} = \frac{Z_2 - Z_1}{2a} = \sin\varphi$$

$$m_z = \cos\phi_{zZ} = \frac{X_2 - X_1}{2a} = \cos\varphi$$

$$\Rightarrow u_2 = D_{2z} m_x, \quad w_2 = D_{2z} m_z$$

In total

$$\begin{bmatrix} u_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} D_{2x} l_x + D_{2z} m_x \\ D_{2x} l_z + D_{2z} m_z \end{bmatrix} = \begin{bmatrix} l_x & m_x \\ l_z & m_z \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2z} \end{bmatrix}$$

With rotation
$(\theta_{2y} = \theta_{2Y})$

$$\begin{bmatrix} u_2 \\ w_2 \\ \theta_{2y} \end{bmatrix} = \begin{bmatrix} l_x & m_x & 0 \\ l_z & m_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2z} \\ \theta_{2Y} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2z} \\ \theta_{2Y} \end{bmatrix}$$

Contributions from the two nodes gives:

$$\begin{bmatrix} u_1 \\ w_1 \\ \theta_{1y} \\ u_2 \\ w_2 \\ \theta_{2y} \end{bmatrix} = \begin{bmatrix} l_x & m_x & 0 & 0 & 0 & 0 \\ l_z & m_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & m_x & 0 \\ 0 & 0 & 0 & l_z & m_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_{1X} \\ D_{1Z} \\ \theta_{1Y} \\ D_{2X} \\ D_{2Z} \\ \theta_{2Y} \end{bmatrix} = \mathbf{T}\mathbf{D}_e; \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_2 \end{bmatrix}$$

2*D* –The transformation matrix is orthogonal

$$\mathbf{T}\,\mathbf{T}^T = \mathbf{T}^T\,\mathbf{T} = \mathbf{I} \qquad \text{unit matrix, dimension } 6 \times 6$$

Equations in global coordinate system (2*D*)

*Element stiffness matrix in **local** coordinate system*

$$\left.\begin{array}{l} \left.\begin{array}{l} \mathbf{d}_e = \mathbf{T}\mathbf{D}_e \\ \mathbf{f}_e = \mathbf{k}_e\mathbf{d}_e \end{array}\right\} \Rightarrow \mathbf{f}_e = \mathbf{k}_e\mathbf{T}\mathbf{D}_e \\ \mathbf{F}_e = \mathbf{T}^T\mathbf{f}_e \end{array}\right\} \Rightarrow \mathbf{F}_e = \underbrace{\mathbf{T}^T\mathbf{k}_e\mathbf{T}}_{\mathbf{K}_e}\,\mathbf{D}_e$$

*Element stiffness matrix in **global** coordinate system*

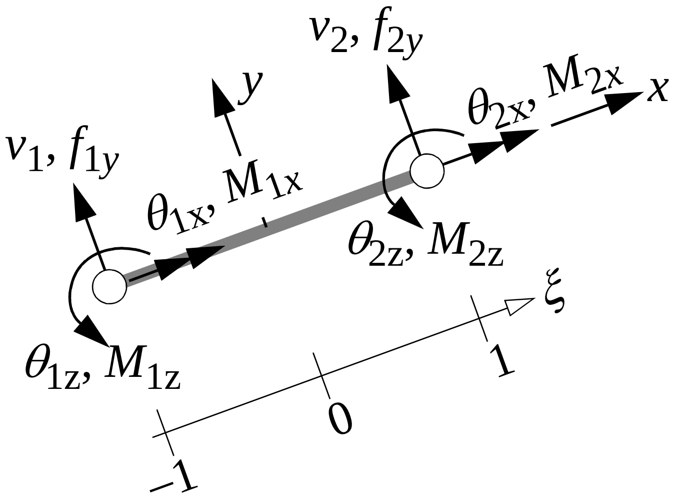
$$\Rightarrow \boxed{\mathbf{F}_e = \mathbf{K}_e\,\mathbf{D}_e}$$

Frames in space (3D)

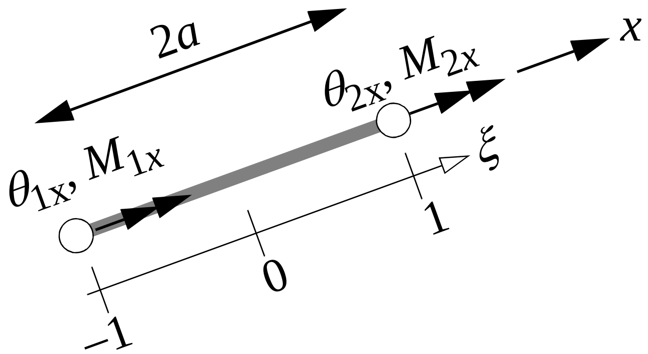
- Assume that the principle axis of the moment of inertia of the beam are oriented along the local *y*- and *z*-axis, respectively. The deflections due to bending $\{w(x), v(x)\}$ are for such a case un-coupled.
- In space, **bending around the z-axis** $\{v_1, \theta_{1z}, v_2, \theta_{2z}\}$ and **torsion around the x-axis** $\{\theta_{1x}, \theta_{2x}\}$ must be considered. Thus 6 D.O.F are added and in total the element contains **12** D.O.F.
- Denote the moments of inertia as I_y and I_z and the polar moment K.

Bending around z-axis:

(analogous with bending around y-axis)



Torsion around the x-axis is un-coupled from all other deformations!



Torsion is analogous with tension!

$$\theta_x(\xi) = \begin{bmatrix} \underbrace{\dfrac{(1-\xi)}{2}}_{N_1^v(\xi)} & \underbrace{\dfrac{(1+\xi)}{2}}_{N_2^v(\xi)} \end{bmatrix} \begin{bmatrix} \theta_{1x} \\ \theta_{2x} \end{bmatrix}$$

Relation between rotation and torque:

$$\theta_{2x} - \theta_{1x} = \frac{2a}{GK} M_{2x}$$

Shear modulus

Polar moment

local element-stiffness matrix

$$\mathbf{k}_e^v = \frac{GK}{2a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

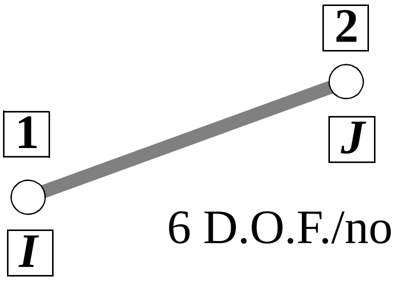
Total element stiffness matrix in the local coord. system (3D)

$$\mathbf{d}_e^T = \begin{bmatrix} u_1 & v_1 & w_1 & \theta_{1x} & \theta_{1y} & \theta_{1z} & u_2 & v_2 & w_2 & \theta_{2x} & \theta_{2y} & \theta_{2z} \end{bmatrix}$$

$$\mathbf{k}_e = \begin{bmatrix}
\frac{EA}{2a} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{2a} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{3EI_z}{2a^3} & 0 & 0 & 0 & \frac{3EI_z}{2a^2} & 0 & -\frac{3EI_z}{2a^3} & 0 & 0 & 0 & -\frac{3EI_z}{2a^2} \\
 & & \frac{3EI_y}{2a^3} & 0 & -\frac{3EI_y}{2a^2} & 0 & 0 & 0 & -\frac{3EI_y}{2a^3} & 0 & -\frac{3EI_y}{2a^2} & 0 \\
 & & & \frac{GK}{2a} & 0 & 0 & 0 & 0 & 0 & -\frac{GK}{2a} & 0 & 0 \\
 & & & & \frac{2EI_y}{a} & 0 & 0 & 0 & -\frac{3EI_y}{2a^2} & 0 & -\frac{2EI_y}{a} & 0 \\
 & & & & & \frac{2EI_z}{a} & 0 & -\frac{3EI_z}{2a^2} & 0 & 0 & 0 & -\frac{2EI_z}{a} \\
 & & & & & & \frac{EA}{2a} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{3EI_z}{2a^3} & 0 & 0 & 0 & \frac{3EI_z}{2a^2} \\
 & & & & & & & & \frac{3EI_y}{2a^3} & 0 & \frac{3EI_y}{2a^2} & 0 \\
 & & \text{SYMMETRIC} & & & & & & & \frac{GK}{2a} & 0 & 0 \\
 & & & & & & & & & & \frac{2EI_y}{a} & 0 \\
 & & & & & & & & & & & \frac{2EI_z}{a}
\end{bmatrix}$$

Transformation: local –> global coordinate system (3*D*)

$$\mathbf{d}_e = \mathbf{T}\mathbf{D}_e$$



6 D.O.F./node

12 D.O.F. in total

$$\mathbf{d}_e = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_{1x} \\ \theta_{1y} \\ \theta_{1z} \\ u_2 \\ v_2 \\ w_2 \\ \theta_{2x} \\ \theta_{2y} \\ \theta_{2z} \end{bmatrix}, \quad \mathbf{D}_e = \begin{bmatrix} D_{1X} \\ D_{1Y} \\ D_{1Z} \\ \theta_{1X} \\ \theta_{1Y} \\ \theta_{1Z} \\ D_{2X} \\ D_{2Y} \\ D_{2Z} \\ \theta_{2X} \\ \theta_{2Y} \\ \theta_{2Z} \end{bmatrix} = \begin{bmatrix} D_{6I-5} \\ D_{6I-4} \\ D_{6I-3} \\ D_{6I-2} \\ D_{6I-1} \\ D_{6I} \\ D_{6J-5} \\ D_{6J-4} \\ D_{6J-3} \\ D_{6J-2} \\ D_{6J-1} \\ D_{6J} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_3 \end{bmatrix}$$

$$\mathbf{T}_3 = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

direction cosines

The 3*D* –transformation matrix is orthogonal

$$\mathbf{T}\,\mathbf{T}^T = \mathbf{T}^T\,\mathbf{T} = \mathbf{I} \qquad \text{unit matrix, dimension } 12 \times 12$$

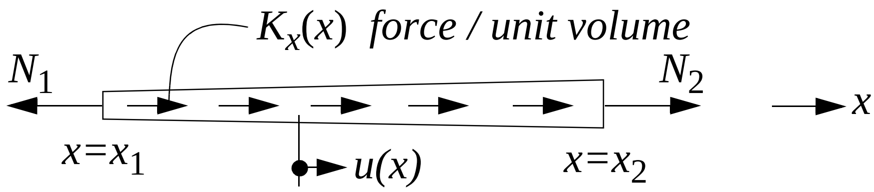
Equations in the global coordinate system (3*D*)

$$\left.\begin{aligned} \mathbf{d}_e &= \mathbf{T}\mathbf{D}_e \\ \mathbf{f}_e &= \mathbf{k}_e\mathbf{d}_e \\ \mathbf{F}_e &= \mathbf{T}^T\mathbf{f}_e \end{aligned}\right\} \Rightarrow \quad \mathbf{F}_e = \mathbf{K}_e\,\mathbf{D}_e$$

$$\mathbf{K}_e = \mathbf{T}^T\mathbf{k}_e\mathbf{T}$$

Lecture 11: FEM for 2D/3D Solids (continuum)

1*D*:

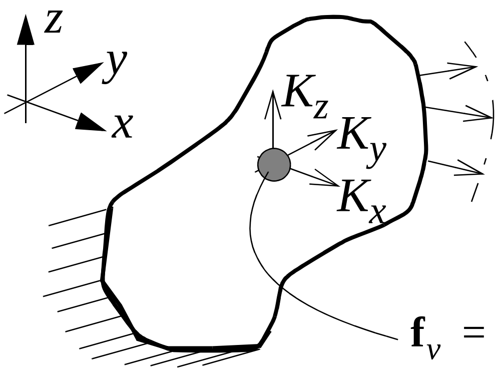


Weak form $\Leftrightarrow$ Principle of Virtual Work

Weight function $v(x) \rightarrow \delta u(x)$ Variation from equilibrium state ("virtual displacement")

$$\underbrace{\int_V \underbrace{\delta\varepsilon\ \sigma}_{\delta W' \text{ (internal work / unit volume)}} \mathrm{d}V}_{\text{Internal virtual work}} = \underbrace{[\delta u\ N]_{x_1}^{x_2} + \int_V \delta u\ K_x \mathrm{d}V}_{\text{External virtual work}}$$

3*D*:



$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Traction vector acting on the surface, S, of the solid [force / unit surface]

$$\mathbf{f}_v = \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}$$

Body force acting in the volume, V, of the solid [force / unit volume]

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

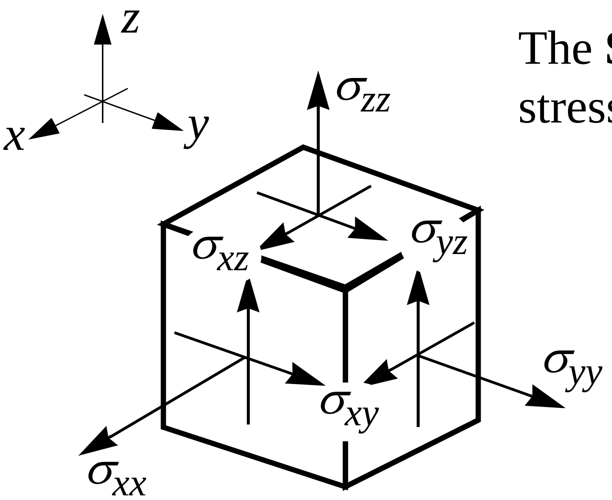
Displacement vector

The quantities $\{\mathbf{u},\ \mathbf{t},\ \mathbf{f}_v\}$ are all functions of x, y, z

Principle of Virtual Work in 3*D*:

$$\underbrace{\int_V \delta W' \mathrm{d}V}_{\textit{Internal Virtual Work}} = \underbrace{\int_S \overbrace{\delta\mathbf{u}^T\,\mathbf{t}}^{\text{scalar products}} \mathrm{d}S + \int_V \overbrace{\delta\mathbf{u}^T\,\mathbf{f}_v}^{\text{scalar products}} \mathrm{d}V}_{\textit{External Virtual Work}}$$

Multi-axial stress- & strain states

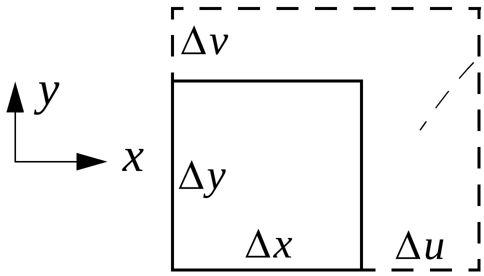


The **Stress matrix, S,** defines the stress state in a material point

$$\mathbf{S} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

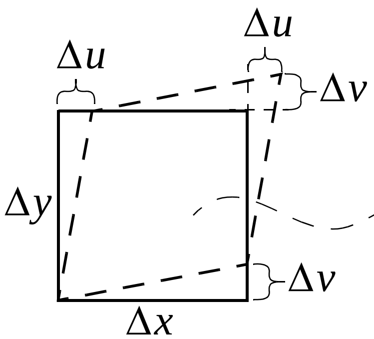
vector form

Normal strains (change of volume):



$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Shear strains (only change of shape):



$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

Stored in vector form: $\boldsymbol{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{xy} & \gamma_{xz} & \gamma_{yz} \end{bmatrix}$

Change (virtual) of internal work / unit volume

$$\delta W' = \delta\varepsilon_{xx}\sigma_{xx} + \delta\varepsilon_{yy}\sigma_{yy} + \delta\varepsilon_{zz}\sigma_{zz} + \delta\gamma_{xy}\sigma_{xy} + \delta\gamma_{xz}\sigma_{xz} + \delta\gamma_{yz}\sigma_{yz}$$

$$\Rightarrow \delta W' = \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}$$

Constitutive relation—linear elastic material

1*D*: $\varepsilon = \frac{\sigma}{E}$ or alternatively $\sigma = E\varepsilon$

Elasticity module (Young's modulus)

3*D*: Example: isotropic linear elastic material

Poisson's ratio

$$\sigma_{xx} = \frac{E}{(1-2\nu)(1+\nu)}[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}] \quad \sigma_{yy} = \ldots \quad \sigma_{zz} = \ldots$$

$$\sigma_{xy} = G\gamma_{xy} \qquad \sigma_{xz} = G\gamma_{xz} \qquad \sigma_{yz} = G\gamma_{yz}$$

Shear modulus

Matrix form:

$$\underbrace{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}}_{\boldsymbol{\sigma}} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}}_{\boldsymbol{\varepsilon}}$$

Elastic stiffness matrix

$$C_{11} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \qquad C_{12} = \frac{E\nu}{(1-2\nu)(1+\nu)}$$

$$\frac{C_{11}-C_{12}}{2} = \frac{E}{2(1+\nu)} = G$$

Change (virtual) of internal work / unit volume

$$\delta W' = \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} = \delta\boldsymbol{\varepsilon}^T\,\mathbf{C}\,\boldsymbol{\varepsilon}$$

Compatibility

(relation between displacements & strains):

Strain:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\mathbf{u}} = \mathbf{L}\ \mathbf{u}$$

Partial
differential operator

Principle of Virtual Work in 3*D* can be formulated as

with $\delta W' = \delta\boldsymbol{\varepsilon}^T\,\mathbf{C}\,\boldsymbol{\varepsilon} = \delta(\mathbf{L}\mathbf{u})^T\mathbf{C}\,(\mathbf{L}\mathbf{u})$

we obtain

$$\underbrace{\int_V \overbrace{\delta(\mathbf{L}\mathbf{u})^T\mathbf{C}\,(\mathbf{L}\mathbf{u})}\mathrm{d}V}_{\textit{Internal Virtual Work}} = \underbrace{\int_S \delta\mathbf{u}^T\,\mathbf{t}\,\mathrm{d}S + \int_V \delta\mathbf{u}^T\,\mathbf{f}_v\,\mathrm{d}V}_{\textit{External Virtual Work}}$$

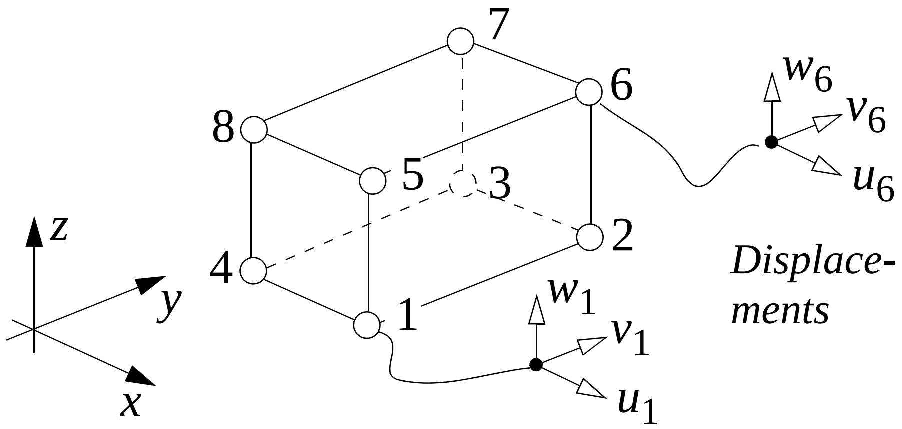
Equilibrium can also be expressed by use of the **L**-operator!

$$\left.\begin{aligned} x\text{-dir.:}\quad & \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} + f_x = 0 \\ y\text{-dir.:}\quad & \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial z} + f_y = 0 \\ z\text{-dir.:}\quad & \frac{\partial\sigma_{zz}}{\partial z} + \frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + f_z = 0 \end{aligned}\right\} \Rightarrow \mathbf{L}^T\boldsymbol{\sigma} + \mathbf{f}_v = 0$$

Approximate displacement interpolation in 2*D*/3*D*

- Divide the solid into volume elements (N_e = number of el.)
- Use "simple" displacement interpolations in each element by use of ***shape functions***
- It is convenient to use the same ***shape functions*** for the displacements in all three directions: u, v and w. If the element has n_d nodes, only n_d different ***shape functions*** are needed

Ex. 3*D* element with 8 nodes ($n_d = 8$)



The displacement in the element can point wise be described by ***the nodal displacements*** and 8 ***shape functions*** as:

$$u(x, y, z) = N_1(x, y, z)u_1 + \ldots + N_8(x, y, z)u_8$$
$$v(x, y, z) = N_1(x, y, z)v_1 + \ldots + N_8(x, y, z)v_8$$
$$w(x, y, z) = N_1(x, y, z)w_1 + \ldots + N_8(x, y, z)w_8$$

Displacement vector on matrix form:

$$\mathbf{u} = \underset{3\times 1}{\begin{bmatrix} u(x,y,z)\\ v(x,y,z)\\ w(x,y,z)\end{bmatrix}} = \underbrace{\begin{array}{c} \begin{array}{ccc}\text{Node 1} & \qquad\qquad & \text{Node 8}\end{array}\\ \left[\begin{array}{ccc:c:ccc} N_1 & 0 & 0 & & N_8 & 0 & 0\\ 0 & N_1 & 0 & \cdots & 0 & N_8 & 0\\ 0 & 0 & N_1 & & 0 & 0 & N_8\end{array}\right]\end{array}}_{\substack{\mathbf{N}\\ 3\times 3n_d \;=\; 3\times 24}} \underbrace{\begin{array}{cl} \left[\begin{array}{c} u_1\\ v_1\\ w_1\\ \vdots\\ u_8\\ v_8\\ w_8\end{array}\right] & \begin{array}{l} \left.\vphantom{\begin{array}{c}u\\v\\w\end{array}}\right\}\text{Node 1 } \mathbf{d}_1\\ \\ \left.\vphantom{\begin{array}{c}u\\v\\w\end{array}}\right\}\text{Node 8 } \mathbf{d}_8\end{array}\end{array}}_{\mathbf{d}_e}$$

Displacement vector of the element → $\mathbf{d}_e$ $\quad 3n_d \times 1 = 24 \times 1$

Strains evaluated from the displ. interpolation:

$$\underbrace{\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}}_{\substack{\boldsymbol{\varepsilon}\\ (6\times 1)}} = \underbrace{\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\end{bmatrix}}_{\substack{\mathbf{L}\\ 6\times 3}} \underbrace{\begin{bmatrix}u\\ v\\ w\end{bmatrix}}_{\substack{\mathbf{u}\\ 3\times 1}} = \mathbf{LNd}_e = \underbrace{\left[\begin{array}{ccc|c|ccc} \multicolumn{3}{c}{\text{Node 1}} & \cdots & \multicolumn{3}{c}{\text{Node 8}} \\ \frac{\partial N_1}{\partial x} & 0 & 0 & & \frac{\partial N_8}{\partial x} & 0 & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0 & & 0 & \frac{\partial N_8}{\partial y} & 0\\ 0 & 0 & \frac{\partial N_1}{\partial z} & \cdots & 0 & 0 & \frac{\partial N_8}{\partial z}\\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & & 0 & \frac{\partial N_8}{\partial z} & \frac{\partial N_8}{\partial y}\\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & & \frac{\partial N_8}{\partial z} & 0 & \frac{\partial N_8}{\partial x}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} & 0\end{array}\right]}_{\substack{\mathbf{B} = \mathbf{LN}\\ 6\times 3n_d \;=\; 6\times 24}} \underbrace{\begin{bmatrix}u_1\\ v_1\\ w_1\\ \vdots\\ u_8\\ v_8\\ w_8\end{bmatrix}}_{\substack{\mathbf{d}_e\\ 3n_d\times 1 \;=\; 24\times 1}}$$

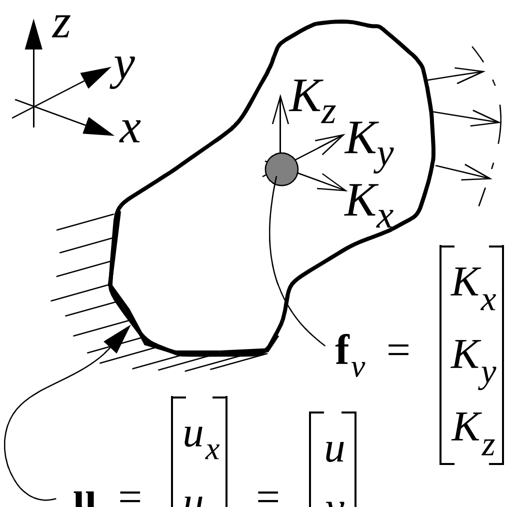
The strains can be expressed on the compact form

$$\boldsymbol{\varepsilon} = \mathbf{Lu} = \mathbf{Bd}_e = \underbrace{\begin{bmatrix}\mathbf{B}_1 & \ldots & \mathbf{B}_8\end{bmatrix}}_{6\times 3}\begin{bmatrix}\overbrace{\mathbf{d}_1}^{3\times 1}\\ \\ \mathbf{d}_8\end{bmatrix} = \underbrace{\mathbf{B}_1\mathbf{d}_1}_{\text{Node 1}} + \ldots + \underbrace{\mathbf{B}_8\mathbf{d}_8}_{\text{Node 8}}$$

A change (virtual) of strains are obtained as

$$\delta\boldsymbol{\varepsilon}^T = \delta(\mathbf{Lu})^T = \delta(\mathbf{Bd}_e)^T = \delta\mathbf{d}_e^T\,\mathbf{B}^T$$

FEM-Eq. derived by the Principle of Virtual Work:



$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \qquad \mathbf{f}_v = \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\overbrace{\int_V \delta\boldsymbol{\varepsilon}^T\, \mathbf{C}\, \boldsymbol{\varepsilon}\, \mathrm{d}V}^{\text{Internal Virtual Work}} = \underbrace{\int_S \delta\mathbf{u}^T\, \mathbf{t}\, \mathrm{d}S + \int_V \delta\mathbf{u}^T\, \mathbf{f}_v \mathrm{d}V}_{\text{External Virtual Work}}$$

FEM-Eq. for one element:

"arbitrary"

$$\delta\mathbf{d}_e^T\left[\int_{V_e} \mathbf{B}^T\mathbf{C}\mathbf{B}\,\mathrm{d}V\right]\mathbf{d}_e = \delta\mathbf{d}_e^T\left[\int_{S_e} \mathbf{N}^T\mathbf{t}\ \mathrm{d}S + \int_{V_e} \mathbf{N}^T\mathbf{f}_v\ \mathrm{d}V\right]$$

$$\Rightarrow \quad \underbrace{\left[\int_{V_e} \mathbf{B}^T\mathbf{C}\mathbf{B}\,\mathrm{d}V\right]}_{\mathbf{k}_e}\mathbf{d}_e = \underbrace{\underbrace{\left[\int_{S_e} \mathbf{N}^T\mathbf{t}\ \mathrm{d}S\right]}_{\mathbf{f}_s} + \underbrace{\left[\int_{V_e} \mathbf{N}^T\mathbf{f}_v\ \mathrm{d}V\right]}_{\mathbf{f}_b}}_{\mathbf{f}_e}$$

$\mathbf{t}$: $\dfrac{\text{force}}{\text{unit surface}}$ $\qquad$ $\mathbf{f}_v$: $\dfrac{\text{force}}{\text{unit volume}}$

Element stiffness matrix $\mathbf{k}_e$

Element load vector $\mathbf{f}_e$

FEM-Eq. for the solid (sum up the contributions from all elements):

E.g. left hand side:

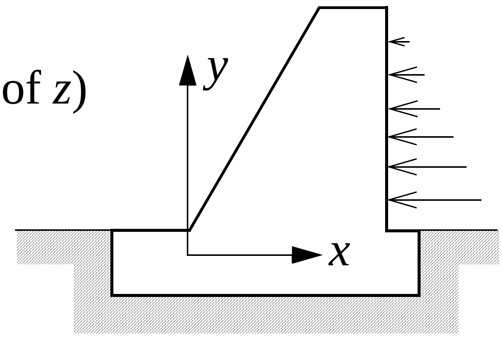
$$\int_V (\ )\mathrm{d}V = \int_{V_1} (\ )\mathrm{d}V + \ldots + \int_{V_{N_e}} (\ )\mathrm{d}V = \sum_{e=1}^{N_e} \mathbf{k}_e = \mathbf{K}$$

Stiffness matrix for the solid

Plane problems (2*D*)

Plane strain (“thick” structures):

$$w = 0, \quad \frac{\partial}{\partial z}(\ ) = 0 \quad \text{(independent of } z\text{)}$$



$$\Rightarrow \varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$$

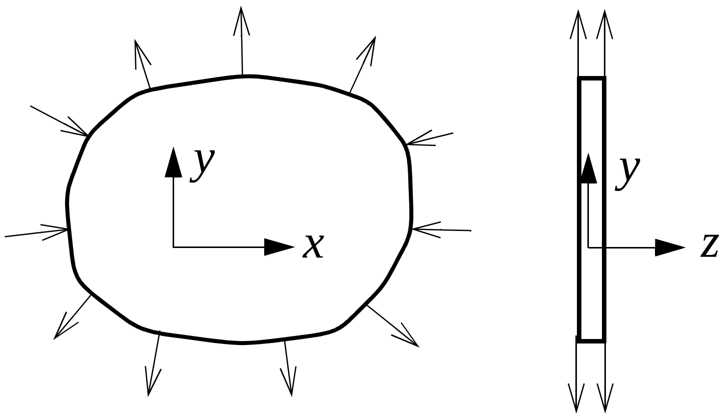
Remove column/row 3, 4 and 5 in **C**

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{(1-\nu)} & 0 \\ \frac{\nu}{(1-\nu)} & 1 & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Plane stress (“thin” structures, e.g. sheet metal):

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

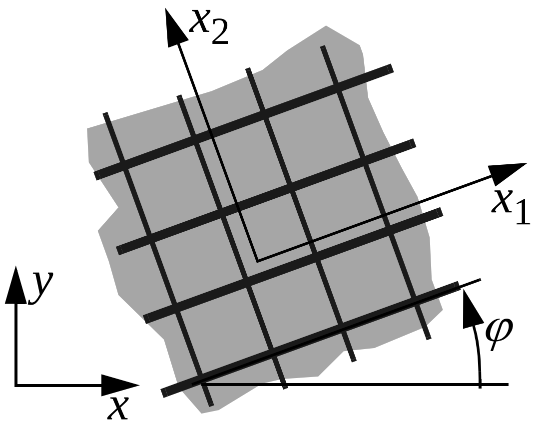
$$\Rightarrow \varepsilon = \mathbf{C}^{-1}\boldsymbol{\sigma}$$



Remove column/row 3, 4 and 5 in $\mathbf{C}^{-1}$. The plane stress elastic stiffness matrix is then obtained as the inverse to the reduced $\mathbf{C}^{-1}$-matrix.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Anisotropic Materials
E.g. Orthotropic material—plane stress (4 mat. par.)



A composite with two sets of fibers orthogonal to each other:

Two different elastic modules in the plane E_1, E_2 and one shear module G_{12}.

One independent parameter that describes the lateral contraction as

$$\nu_{12}/E_1 = \nu_{21}/E_2 .$$

(5 additional parameters in 3D: E_3, G_{13}, G_{23} and two contraction parameters)

Description in the local coordinate system x_1–x_2:

$$\bar{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \bar{\mathbf{S}}\bar{\sigma}$$

$$\Rightarrow \quad \bar{\sigma} = \bar{\mathbf{C}}\bar{\varepsilon} \qquad \text{where} \quad \bar{\mathbf{C}} = \bar{\mathbf{S}}^{-1} = \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Transformation to the global coordinate system x–y:

$$\left.\begin{array}{l} \bar{\sigma} = \mathbf{L}\sigma \quad \Leftrightarrow \quad \sigma = \mathbf{L}^{-1}\bar{\sigma} \\ \bar{\varepsilon} = \mathbf{L}^{-T}\varepsilon \end{array}\right\} \Rightarrow \sigma = \mathbf{L}^{-1}\bar{\sigma} = \mathbf{L}^{-1}\bar{\mathbf{C}}\bar{\varepsilon} = \underbrace{\mathbf{L}^{-1}\bar{\mathbf{C}}\mathbf{L}^{-T}}\varepsilon = \mathbf{C}\varepsilon$$

Used in FEM

where

$$\mathbf{L} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2-s^2 \end{bmatrix} \Leftrightarrow \mathbf{L}^{-1} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2-s^2 \end{bmatrix} \quad \begin{array}{l} c = \cos\varphi \\ s = \sin\varphi \end{array}$$

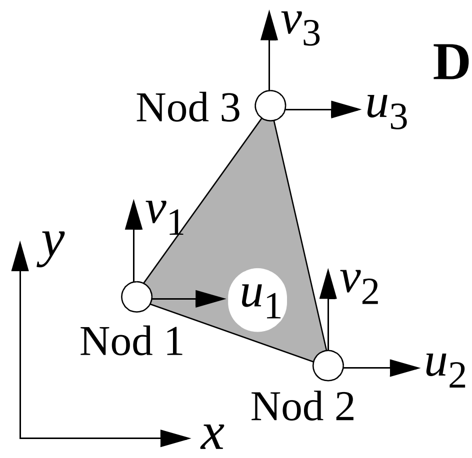
Eqs. 1.17, 1.18, 2.21 & 2.22 in H.S.

Note! In a FE-analysis, also the principal material orientation, φ, must be given as input in addition the material parameters: E_1, E_2, G_{12} & ν_{12}

Lecture 12

FEM-elements for plane problems (2*D*)

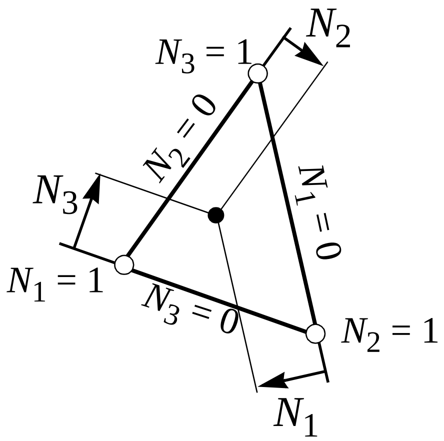
"<u>C</u>onstant <u>S</u>train <u>T</u>riangle"-Element



Displacement interpolation (linear):

$$\mathbf{u} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}}_{\mathbf{d}_e}$$

Shape functions:



$$N_1 = \frac{1}{2A_e}[y_{23}(x - x_2) + x_{32}(y - y_2)]$$

$$N_2 = \frac{1}{2A_e}[y_{31}(x - x_3) + x_{13}(y - y_3)]$$

$$N_3 = \frac{1}{2A_e}[y_{12}(x - x_1) + x_{21}(y - y_1)]$$

where A_e = area of element

and $x_{kl} = x_k - x_l \qquad y_{kl} = y_k - y_l$

Strain: $$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \mathbf{Lu} = \mathbf{LNd}_e = \mathbf{Bd}_e$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} \end{bmatrix} = \frac{1}{2A_e} \begin{array}{c} \begin{array}{ccc} \textit{Nod 1} & \textit{Nod 2} & \textit{Nod 3} \\ \overbrace{\quad\mathbf{B}_1\quad} & \overbrace{\quad\mathbf{B}_2\quad} & \overbrace{\quad\mathbf{B}_3\quad} \end{array} \\ \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \end{array}$$

Note! constant matrix!

Element matrices/vectors: CST-element

Element stiffness matrix:

thickness

$$\mathbf{k}_e = \int_{V_e} \mathbf{B}^T\mathbf{C}\mathbf{B}\,\mathrm{d}V = hA_e\mathbf{B}^T\mathbf{C}\mathbf{B} = hA_e\begin{bmatrix} \mathbf{B}_1^T\mathbf{C}\mathbf{B}_1 & \mathbf{B}_1^T\mathbf{C}\mathbf{B}_2 & \mathbf{B}_1^T\mathbf{C}\mathbf{B}_3 \\ \mathbf{B}_2^T\mathbf{C}\mathbf{B}_1 & \mathbf{B}_2^T\mathbf{C}\mathbf{B}_2 & \mathbf{B}_2^T\mathbf{C}\mathbf{B}_3 \\ \mathbf{B}_3^T\mathbf{C}\mathbf{B}_1 & \mathbf{B}_3^T\mathbf{C}\mathbf{B}_2 & \mathbf{B}_3^T\mathbf{C}\mathbf{B}_3 \end{bmatrix}$$

6×6

symmetric!

2×2

Elastic stiffness matrices in 2D

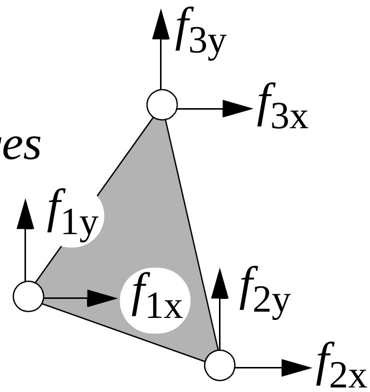
$$\mathbf{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{(1-\nu)} & 0 \\ \dfrac{\nu}{(1-\nu)} & 1 & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2(1-\nu)} \end{bmatrix} \qquad \mathbf{C} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}$$

Plane strain ($w = 0$)

Plane stress ($\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$)

External forces:

=> *Calculate consistent nodal forces*



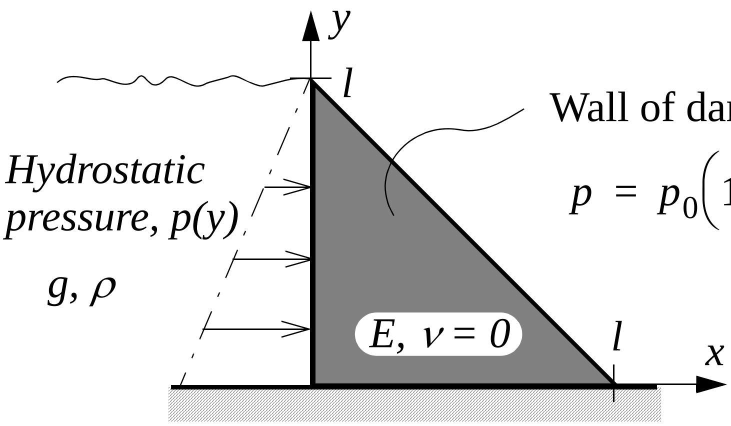
$$\mathbf{f}_e = \mathbf{f}_p + \mathbf{f}_s + \mathbf{f}_b$$

Point forces acting in nodes

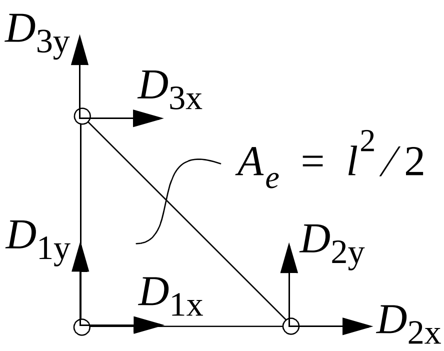
$$\frac{\textit{force}}{\textit{unit surface}} \qquad \frac{\textit{force}}{\textit{unit volume}}$$

$$\mathbf{f}_p = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{bmatrix}_{\text{Point force}} \qquad \mathbf{f}_s = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{bmatrix}_s = \int_{S_e} \mathbf{N}^T\mathbf{t}\,\mathrm{d}S \qquad \mathbf{f}_b = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{bmatrix}_b = \int_{V_e} \mathbf{N}^T\mathbf{f}_v\,\mathrm{d}V$$

Example: FEM-analysis with one CST-element



$$p = p_0\left(1 - \frac{y}{l}\right) \quad \text{where} \quad p_0 = g\rho l$$



<u>FEM-analysis:</u>

Boundary Cond.: $D_{1x} = D_{1y} = D_{2x} = D_{2y} = 0$

=> Reaction forces: R_{1x}, R_{1y}, R_{2x}, R_{2y}

Shape functions: $N_1 = 1 - \frac{x}{l} - \frac{y}{l};\ N_2 = \frac{x}{l};\ N_3 = \frac{y}{l}$

B-matrix:

$$B = \frac{1}{l}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

"Elastic stiffness matrix": *(assume plane strain)*

$$\mathbf{C} = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

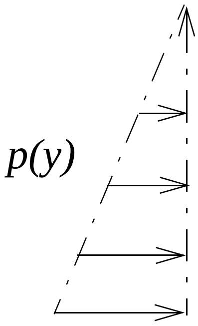
Element stiffness matrix:

$$\mathbf{k}_e = \mathbf{B}^T\mathbf{C}\mathbf{B}hA_e = \frac{hl^2}{2}\begin{bmatrix} \boldsymbol{B}_1^T \\ \boldsymbol{B}_2^T \\ \boldsymbol{B}_3^T \end{bmatrix}\mathbf{C}\begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 \end{bmatrix} =$$

$$= \frac{Eh}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

cont. CST-example

Load vector:



=> Traction vector: $\mathbf{t} = \begin{bmatrix} p \\ 0 \end{bmatrix} = p_0 \begin{bmatrix} 1 - \dfrac{y}{l} \\ 0 \end{bmatrix}$

Acting on the surface at $x = 0$, i.e. between Nodes 1 & 3

where $N_1 = 1 - y/l$; $N_2 = 0$; $N_3 = y/l$

$$\Rightarrow \mathbf{f}_s = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{bmatrix} = \int_0^l [\mathbf{N}^T\mathbf{t}]_{x=0} \overbrace{h\,\mathrm{d}y}^{\text{surface element}} = \int_0^l \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}_{x=0} p_0 \begin{bmatrix} 1 - \dfrac{y}{l} \\ 0 \end{bmatrix} h\,\mathrm{d}y = \frac{p_0 h l}{6} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Force / unit volume: $\mathbf{f}_b = \mathbf{0}$*, since the volume forces (K_x, K_y) are assumed to be zero!*

Equation system:

$$\frac{Eh}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} D_{1x} \\ D_{1y} \\ D_{2x} \\ D_{2y} \\ D_{3x} \\ D_{3y} \end{bmatrix} = \begin{bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 0 \\ 0 \end{bmatrix} + \frac{p_0 h l}{6} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solve equations (5) & (6):

$$\frac{Eh}{4} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} D_{3x} \\ D_{3y} \end{bmatrix} = \frac{p_0 h l}{6} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} D_{3x} \\ D_{3y} \end{bmatrix} = \frac{2}{3} \frac{p_0}{E} l \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

cont. CST-example

Post-processing:

Reaction forces (obtained from Eqs. 1-4):

Eq. (1): $R_{1x} = \frac{Eh}{4}(-D_{3x}) - \frac{2p_0hl}{6} = -\frac{p_0hl}{2}$

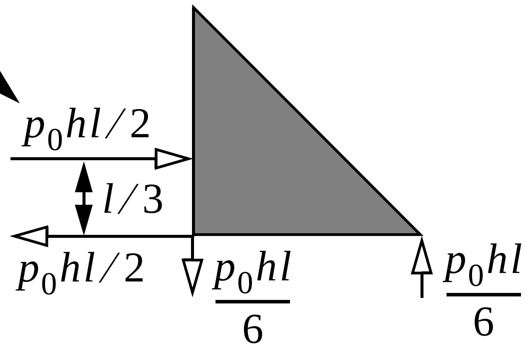
Eq. (2): $R_{1y} = \frac{Eh}{4}(-D_{3x}) = -\frac{p_0hl}{6}$

Eq. (3): $R_{2x} = 0$

Eq. (4): $R_{2y} = \frac{Eh}{4}D_{3x} = \frac{p_0hl}{6}$

distributed load with resultant located l/3 from the bottom

Note that global equilibrium is satisfied!



Stresses:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \mathbf{C}\boldsymbol{\varepsilon} = \mathbf{CBd}_e = \mathbf{C}\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \end{bmatrix}\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix} = \begin{Bmatrix} \mathbf{d}_1 = \mathbf{0} \\ \mathbf{d}_2 = \mathbf{0} \end{Bmatrix} =$$

$$= \mathbf{CB}_3\mathbf{d}_3 = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\frac{1}{l}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{2}{3}\frac{p_0}{E}l\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{p_0}{3}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

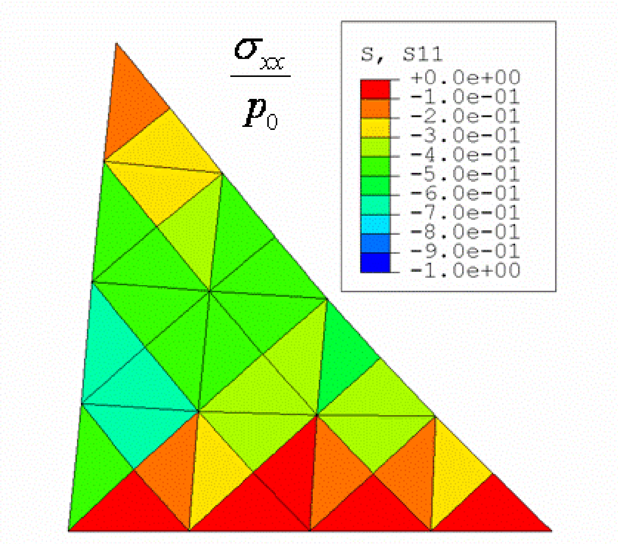
only shear stresses!

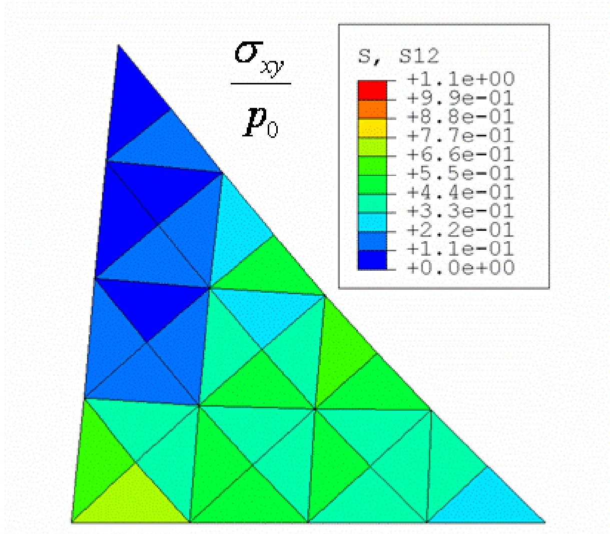
This solution is far from the exact solution, why?

Results from FEM analysis with ABAQUS

32 linear triangular 3-node elements

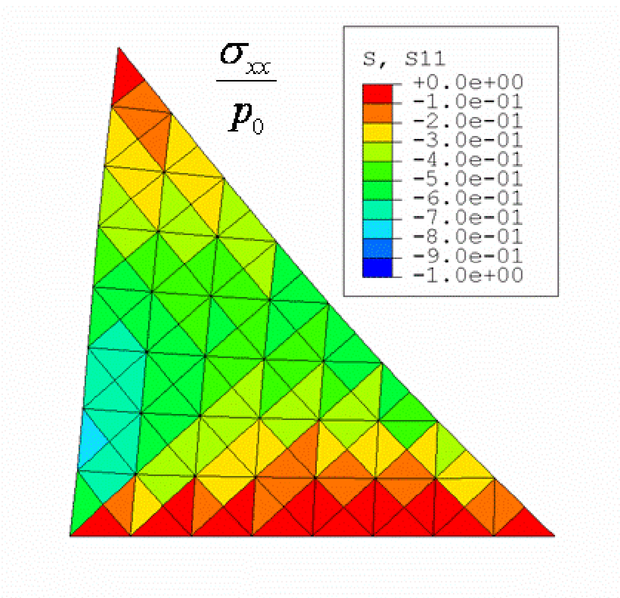
"1 element solution": $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{xy} = 0.33p_0$

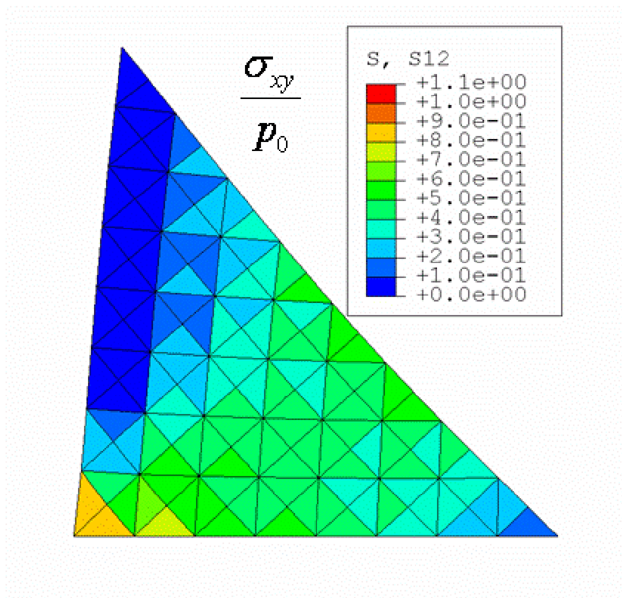




128 linear triangular 3-node elements

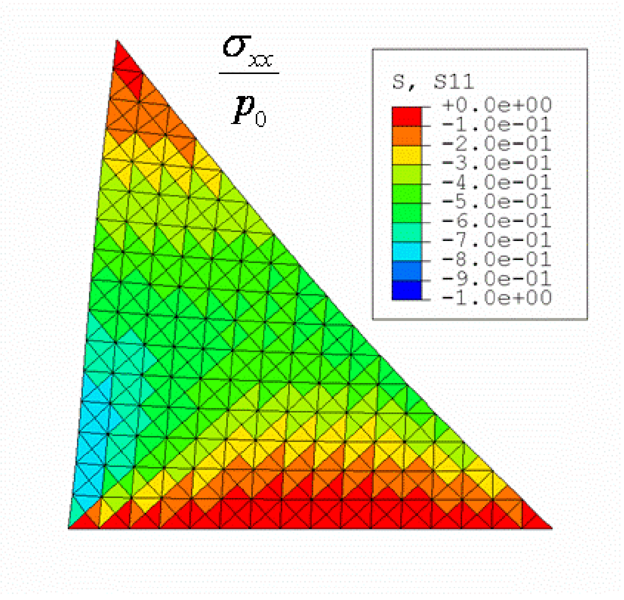
"1 element solution": $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{xy} = 0.33p_0$

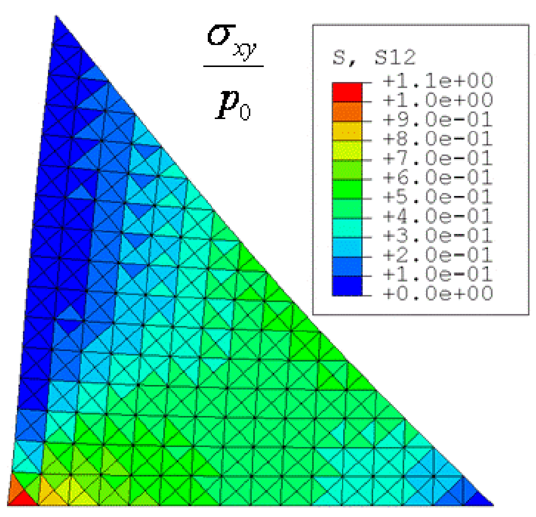




512 linear triangular 3-node elements

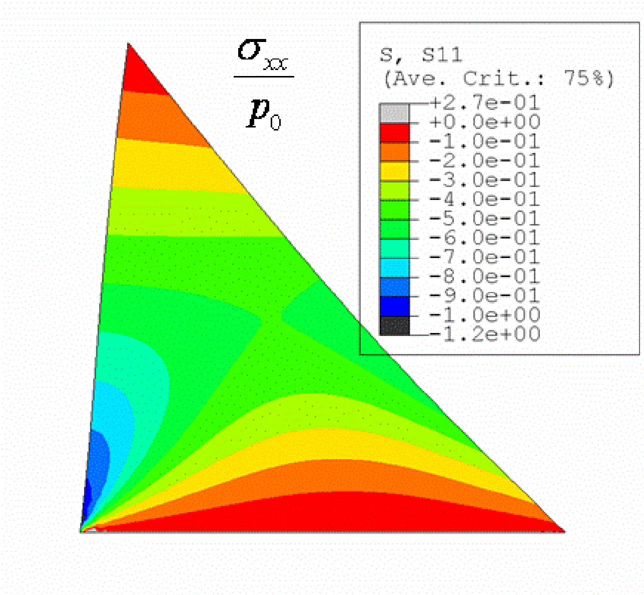
"1 element solution": $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{xy} = 0.33p_0$

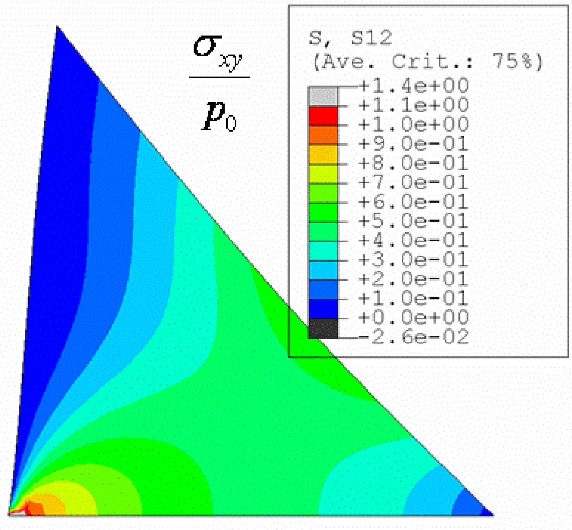




1936 quadratic triangular 6-node elements

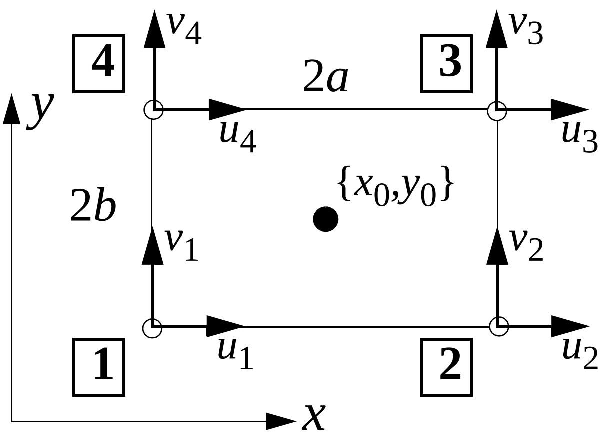
" 1 element solution": $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{xy} = 0.33p_0$



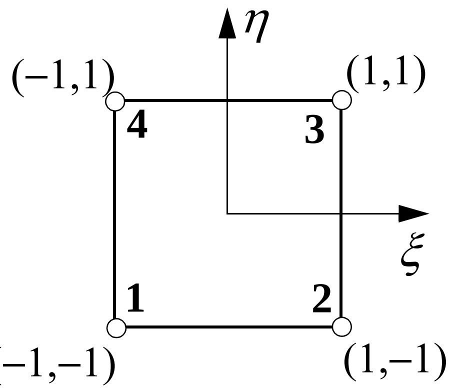


Lecture 13
Plane element (2*D*) with 4 nodes

Bi-linear Rectangular element:



$\Leftrightarrow$



$$\xi = (x - x_0)/a \qquad \eta = (y - y_0)/b$$

(natural coordinates)

Displacement interpolation (bi-linear):

$$\mathbf{u} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & \cdots & N_4 & 0 \\ 0 & N_1 & & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix} = \mathbf{N}\mathbf{d}_e$$

Shape functions:

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta) \qquad N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

Strain:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \mathbf{N}\mathbf{d}_e = \mathbf{B}\mathbf{d}_e$$

$$\Rightarrow \mathbf{B} = \begin{array}{c} \begin{array}{ccc} \overbrace{\qquad\qquad}^{\text{Node 1}\ \mathbf{B}_1} & \qquad & \overbrace{\qquad\qquad}^{\text{Node 4}\ \mathbf{B}_4} \end{array} \\ \begin{bmatrix} \dfrac{N_{1,\xi}}{a} & 0 & \cdots & \dfrac{N_{4,\xi}}{a} & \\ 0 & \dfrac{N_{1,\eta}}{b} & \cdots & & \dfrac{N_{4,\eta}}{b} \\ \dfrac{N_{1,\eta}}{b} & \dfrac{N_{1,\xi}}{a} & \cdots & \dfrac{N_{4,\eta}}{b} & \dfrac{N_{4,\xi}}{a} \end{bmatrix} \end{array}$$

cont. Bi-linear Rectangle

Partial derivatives:

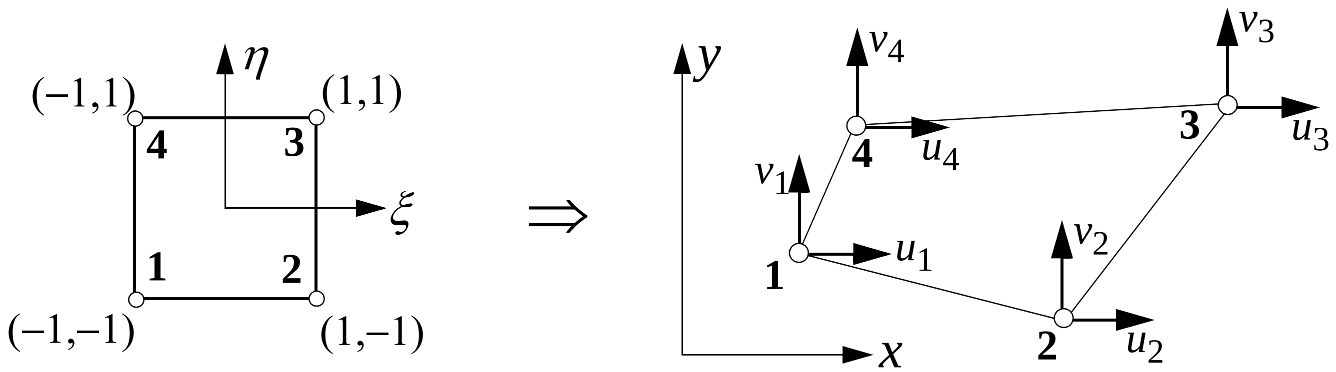
$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{a}\frac{\partial N_i}{\partial \xi} = \frac{1}{a}N_{i,\,\xi} \qquad \frac{\partial N_i}{\partial y} = \ldots = \frac{1}{b}N_{i,\,\eta}$$

where

$$N_{1,\,\xi} = -\frac{(1-\eta)}{4} \quad N_{1,\,\eta} = -\frac{(1-\xi)}{4} \quad N_{2,\,\xi} = \frac{(1-\eta)}{4} \quad N_{2,\,\eta} = -\frac{(1+\xi)}{4}$$

$$N_{3,\,\xi} = \frac{(1+\eta)}{4} \quad N_{3,\,\eta} = \frac{(1+\xi)}{4} \quad N_{4,\,\xi} = -\frac{(1+\eta)}{4} \quad N_{4,\,\eta} = \frac{(1-\xi)}{4}$$

Generalization: a bi-linear quadrilateral element



Coordinate transformation:

$$x = x(\xi, \eta) = N_1 x_1 + \ldots + N_4 x_4 = \sum_{i=1}^{4} N_i x_i$$

$$y = y(\xi, \eta) = N_1 y_1 + \ldots + N_4 y_4 = \sum_{i=1}^{4} N_i y_i$$

(Same shape functions as in the 4-node rectangular element)

The element is called ***isoparametric***, since the same ***interpolation*** is used to describe both geometry (x, y) and displacements (u, v)

cont. isoparametric bi-linear element

Strain: $$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \mathbf{N}\mathbf{d}_e = \mathbf{B}\mathbf{d}_e$$

The **B**-matrix can be divided into 4 sub-matrices: $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}$

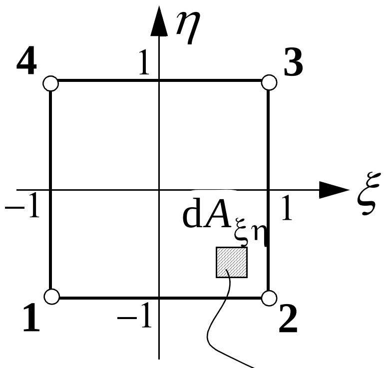
where each sub-matrix $\mathbf{B}_i$ is defined as $\mathbf{B}_i = \begin{bmatrix} \partial N_i/\partial x & 0 \\ 0 & \partial N_i/\partial y \\ \partial N_i/\partial y & \partial N_i/\partial x \end{bmatrix}$

the partial derivatives in $\mathbf{B}_i$ are given by $\begin{bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{bmatrix}$

where **J** is the ***Jacobi matrix*** of the coordinate transformation defined as

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{i=1}^{4} N_{i,\xi}\, x_i & \displaystyle\sum_{i=1}^{4} N_{i,\xi}\, y_i \\ \displaystyle\sum_{i=1}^{4} N_{i,\eta}\, x_i & \displaystyle\sum_{i=1}^{4} N_{i,\eta}\, y_i \end{bmatrix}$$

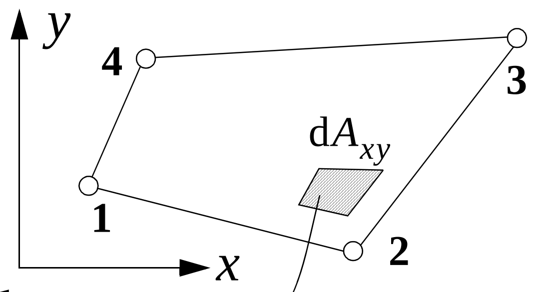
Summary: Isoparametric quadrilateral bi-linear element (2D)



$\Rightarrow$

Area relation:

$|\mathbf{J}|\mathrm{d}A_{\xi\eta} = \mathrm{d}A_{xy}$



Coordinate transformation: $x(\xi,\eta) = \sum_{i=1}^{4} N_i x_i \quad y(\xi,\eta) = \sum_{i=1}^{4} N_i y_i$

Partial derivatives (compact notation): $N_{i,x} = \partial N_i / \partial x \quad N_{i,\eta} = \partial N_i / \partial \eta$

$$\begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{bmatrix} \text{ where } \mathbf{J} = \begin{bmatrix} x,_{\xi} & y,_{\xi} \\ x,_{\eta} & y,_{\eta} \end{bmatrix} = \begin{bmatrix} \sum x_i N_{i,\xi} & \sum y_i N_{i,\xi} \\ \sum x_i N_{i,\eta} & \sum y_i N_{i,\eta} \end{bmatrix}$$

part of the **B**-*matrix*

Element stiffness matrix: $\mathbf{k}_e = h \int_{A_e} \mathbf{B}^T \mathbf{C} \mathbf{B} \mathrm{d}A = h \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T \mathbf{C} \mathbf{B}\, |\mathbf{J}| \mathrm{d}\xi \mathrm{d}\eta$

Element load vectors:

$\frac{\textit{force}}{\textit{unit surface}}$ $$\mathbf{f}_s = \int_{S_e} \mathbf{N}^T \mathbf{t} \mathrm{d}S = \int_{-1}^{1} (\mathbf{N}^T\mathbf{t})\Big|_{\eta=-1} h l_{12} \mathrm{d}\xi + \int_{-1}^{1} (\mathbf{N}^T\mathbf{t})\Big|_{\xi=1} h l_{23} \mathrm{d}\eta$$
$$+ \int_{-1}^{1} (\mathbf{N}^T\mathbf{t})\Big|_{\eta=1} h l_{34} \mathrm{d}\xi + \int_{-1}^{1} (\mathbf{N}^T\mathbf{t})\Big|_{\xi=-1} h l_{41} \mathrm{d}\eta$$

where e.g. $l_{12} = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\frac{\textit{force}}{\textit{unit volume}}$ $$\mathbf{f}_b = \int_{V_e} \mathbf{N}^T \begin{bmatrix} K_x \\ K_y \end{bmatrix} \mathrm{d}V = \int_{-1}^{1}\int_{-1}^{1} \mathbf{N}^T \begin{bmatrix} K_x \\ K_y \end{bmatrix} h |\mathbf{J}| \mathrm{d}\xi \mathrm{d}\eta$$

Lecture 14 & 15

1. Isoparametric quadrilateral elements
   => Repetition + Example 6.7

2. Numerical integration

3. Higher order 2D-elements

4. Elements for 3*D* solids

5. Compatibility, symmetry, boundary conditions, etc. (from the text book, chap. 11)

6. Convergence & sources of error in FEM

7. Static condensation & substructures

8. Constraint equations

1. Repetition: Coordinate transformation in an isoparametric element

The same interpolation is used to describe both ***geometry (x, y)*** and ***displacement (u, v)*** in an isoparametric element, thus

Geometry: $$x(\xi,\eta) = \sum_{i=1}^{n} N_i x_i \qquad y(\xi,\eta) = \sum_{i=1}^{n} N_i y_i$$

Displacement: $$u(\xi,\eta) = \sum_{i=1}^{n} N_i u_i \qquad v(\xi,\eta) = \sum_{i=1}^{n} N_i v_i$$

Deformation (strain): $$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \mathbf{N}\mathbf{d}_e = \mathbf{B}\mathbf{d}_e$$

Partial derivatives of $N_i(x(\xi,\eta);y(\xi,\eta))$ w.r.t. x and y is derived by use of the ***chain rule***

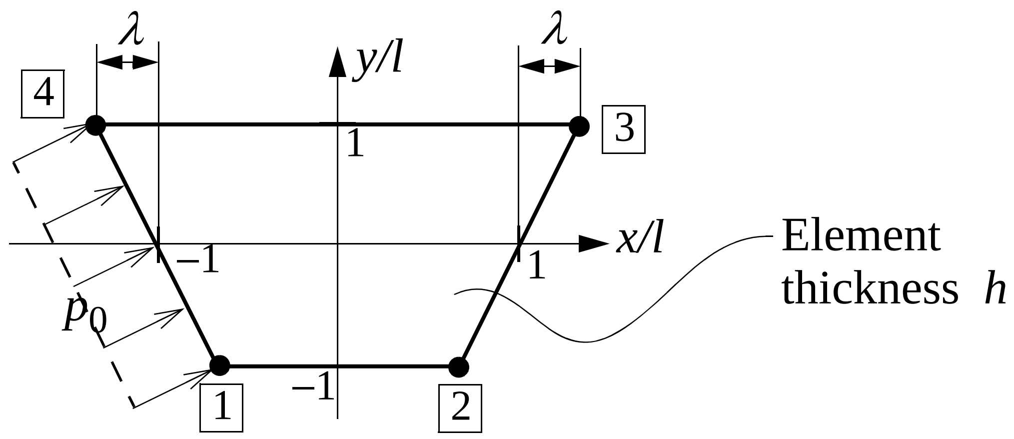
$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial N_i}{\partial y}\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial N_i}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial N_i}{\partial y}\dfrac{\partial y}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{\mathbf{J}\ \text{“Jacobi matrix”}} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

(Note! it is **not** a symmetric matrix)

Examples 6.7 & 6.8

Isoparametric quadrilateral element



Determine

6.7(a) Coordinate transformation: $x = x(\xi,\eta)$ & $y = y(\xi,\eta)$

6.7(b) Jacobi matrix **J** and its determinant $|\mathbf{J}|$

6.7(variant of d) The sub-matrix $\mathbf{B}_1$ of the B-matrix

6.8(a) Contribution p_0 to the element load vector $\mathbf{f}_e$

2. Numerical integration

1D: $$I = \int_L f(x)\mathrm{d}x = \int_{-1}^{1} \underbrace{f(\xi)|\mathbf{J}|}\mathrm{d}\xi = \sum_{i=1}^{m_\xi} F(\xi)w_i$$

2D: $$I = \int_A f(x, y)\mathrm{d}A = \int_{-1}^{1}\int_{-1}^{1} \underbrace{f(\xi, \eta)|\mathbf{J}|}\mathrm{d}\xi\mathrm{d}\eta = \sum_{i=1}^{m_\xi}\sum_{j=1}^{m_\eta} F(\xi, \eta)w_i w_j$$

3D: $$I = \int_V f(x, y, z)\mathrm{d}A = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi, \underbrace{\eta, \zeta)|\mathbf{J}|\mathrm{d}\xi\mathrm{d}\eta}\mathrm{d}\zeta =$$

$$= \sum_{i=1}^{m_\xi}\sum_{j=1}^{m_\eta}\sum_{k=1}^{m_\zeta} F(\xi, \eta)w_i w_j w_k$$

Taken from "The finite element method", G.R. Liu & S.S. Quek

146 CHAPTER 7 FEM FOR TWO-DIMENSIONAL SOLIDS

Table 7.1. Gauss integration points and weight coefficients

m	ξ_j	w_j	Accuracy n
1	0	2	1
2	$-1/\sqrt{3}$, $1/\sqrt{3}$	1,1	3
3	$-\sqrt{0.6}$, 0, $\sqrt{0.6}$	5/9, 8/9, 5/9	5
4	−0.861136, −0.339981, 0.339981, 0.861136	0.347855, 0.652145, 0.652145, 0.347855	7
5	−0.906180, −0.538469, 0, 0.538469, 0.906180	0.236927, 0.478629, 0.568889, 0.478629, 0.236927	9
6	−0.932470, −0.661209, −0.238619, 0.238619, 0.661209, 0.932470	0.171324, 0.360762, 0.467914, 0.467914, 0.360762, 0.171324	11

3. Higher order 2*D*-elements

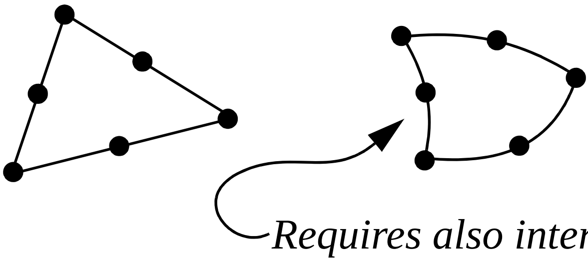
- Displacement interpolation 2nd order polynomial or higher
- Allows for modeling of curved boundaries

Triangular elements (, 3-sides, text book. pp. 153-156):

Shape functions are based on ***base functions*** derived from
Pascal's triangle => complete polynomials

E.g. quadratic interpolation: *base functions* = $\{1 \quad x \quad y \quad x^2 \quad xy \quad y^2\}$

=> 6 nodes
Shape fcn., can be expressed by area coordinates



Requires also interpolation of x- & y-coordinates

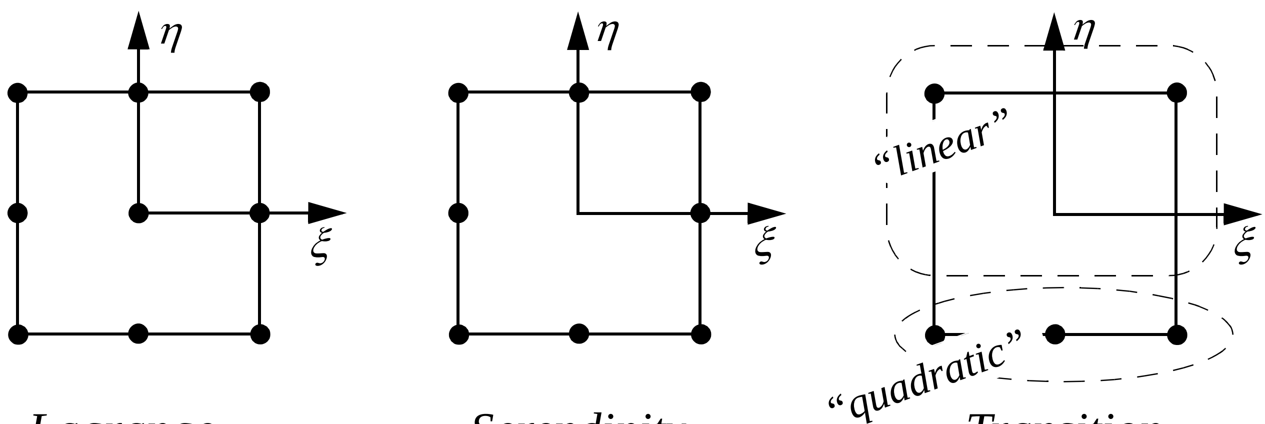
Quadrilateral elements (4-sides, text book pp. 156-160):

Isoparametric: same *interpolation* for **x** (geom.) and **u** (displ.)

Shape functions expressed using natural coordinates: $\left\{ -1 \le \begin{matrix}\xi \\ \eta\end{matrix} \le 1 \right\}$

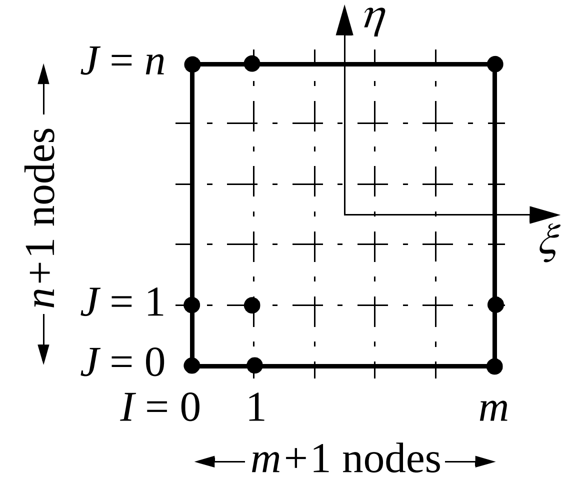
Different types of elements:

- Lagrange (full Lagrange interpolation in each direction: ξ, η)
- Serendipity (internal nodes removed from Lagrange el.)
- Transition elements (e.g. linear — quadratic interpolation)



Lagrange *Serendipity* *Transition*

Example: Element of Lagrange type



The shape function for node i is given by (the index I & J determines the node number)

$$N_i = l_I^m(\xi)\, l_J^n(\eta)$$

Lagrange interpolants where m & n defines the degree of the polynomial

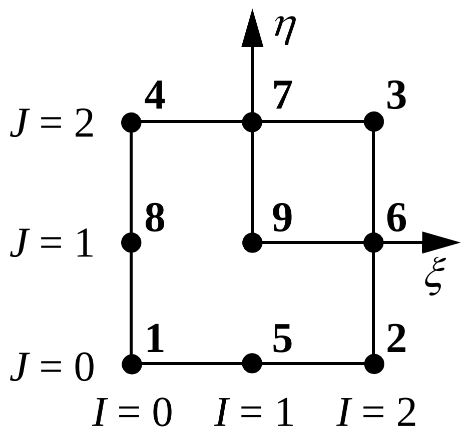
Definition: Lagrange interpolants (text book p. 87)

$$l_k^n(x) = \prod_{\substack{i=0\\ i\neq k}}^{i=n} \frac{(x-x_i)}{(x_k-x_i)} = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

Example: quadratic interpolation in ξ- & η-direction

node pos.

$$\Rightarrow m = n = 2;\ \xi_I = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \ \&\ \ \eta_J = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$



$$\Rightarrow \quad l_0^2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_0-\xi_1)(\xi_0-\xi_2)} =$$

$$= \frac{(\xi-0)(\xi-1)}{(-1-0)(-1-1)} = \frac{\xi(\xi-1)}{2}$$

$$l_1^2(\xi) = \frac{(\xi+1)(\xi-1)}{(0+1)(0-1)} = 1-\xi^2$$

The shape functions becomes:

$$l_2^2(\xi) = \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} = \frac{\xi(\xi+1)}{2}$$

$$N_1 = l_0^2(\xi)\, l_0^2(\eta) = \tfrac{1}{4}\xi(\xi-1)\eta(\eta-1)$$

$$\ddots$$

$$N_5 = l_1^2(\xi)\, l_0^2(\eta) = \tfrac{1}{2}(1-\xi^2)\eta(\eta-1)$$

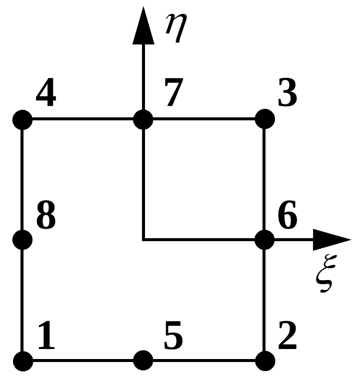
$$\ddots$$

$$N_9 = l_1^2(\xi)\, l_1^2(\eta) = (1-\xi^2)(1-\eta^2)$$

Example: Elements of Serendipity type

- Internal nodes removed from a Lagrange element
- Non-complete Lagrange interpolation
- Shape functions are derived from "inspection" and use of the properties shape functions must have

Example: Serendipity element based on a quadratic Lagrange element, where the centre node (node 9) is removed



For instance $N_1(\xi,\eta)$ must satisfy:

$N_1 = 0$:

(i) at nodes 2, 3 & 6, where $1-\xi = 0$

(ii) at nodes 3, 4 & 7, where $1-\eta = 0$

(iii) at nodes 8 & 5, where $-\xi-\eta-1 = 0$

$N_1 = 1$: at node 1, where $\xi = \eta = -1$

Construct shape functions by use of (i)-(iii)

(same idea as for Lagrange interpolation)

$$\Rightarrow N_1 = c_1(1-\xi)(1-\eta)(-\xi-\eta-1)$$

The requirement: $N_1(\xi = \eta = -1) = 1$ then gives $c_1 = \frac{1}{4}$

The shape functions become (see text book p. 158):

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$\vdots$$

$$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

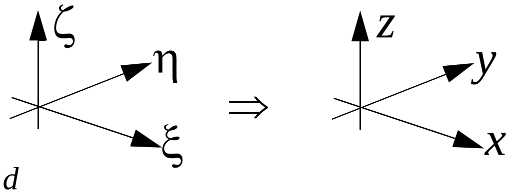
$$\vdots$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

4. Elements for 3*D* solids

- Same principles as for elements for 2*D* solids
- In general ***isoparametric*** elements are used (the same ***shape functions*** are used for interpolation of displacements and geometry)

Coordinate transformation (geometry):



number of nodes in the element

$$x = \sum_{i=1}^{n_d} N_i x_i \qquad y = \sum_{i=1}^{n_d} N_i y_i \qquad z = \sum_{i=1}^{n_d} N_i z_i$$

Displacements:

$$\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{N}\mathbf{d}_e = \begin{bmatrix} N_1 & 0 & 0 & \dots & N_{nd} & 0 & 0 \\ 0 & N_1 & 0 & \dots & 0 & N_{nd} & 0 \\ 0 & 0 & N_1 & \dots & 0 & 0 & N_{nd} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_{nd} \\ v_{nd} \\ w_{nd} \end{bmatrix}$$

Node 1

Node n_d

Strains:

B-matrix
$(6 \times 3n_d)$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \mathbf{L}\mathbf{u} = \mathbf{L}\,\mathbf{N}\,\mathbf{d}_e = \mathbf{B}\mathbf{d}_e$$

Partial derivatives of shape functions

$$\begin{bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \\ \partial N_i/\partial z \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \\ \partial N_i/\partial \zeta \end{bmatrix}$$

Jacobi matrix:

$$\mathbf{J} = \begin{bmatrix} x_{,\xi} & y_{,\xi} & z_{,\xi} \\ x_{,\eta} & y_{,\eta} & z_{,\eta} \\ x_{,\zeta} & y_{,\zeta} & z_{,\zeta} \end{bmatrix} = \begin{bmatrix} N_{1,\xi} & \dots & N_{nd,\xi} \\ N_{1,\eta} & \dots & N_{nd,\eta} \\ N_{1,\zeta} & \dots & N_{nd,\zeta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_{nd} & y_{nd} & z_{nd} \end{bmatrix}$$

Element stiffness matrix:

$$\mathbf{k}_e = \int_{V_e} \mathbf{B}^T\mathbf{C}\mathbf{B}\,\mathrm{d}V = \iiint \mathbf{B}^T\mathbf{C}\mathbf{B}\,|\mathbf{J}|\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

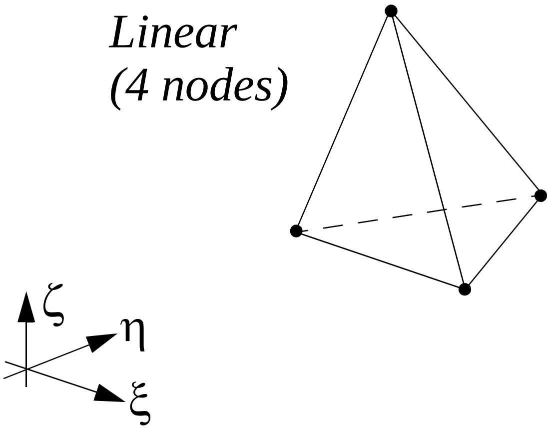
Volume element

Examples of element types in 3*D*

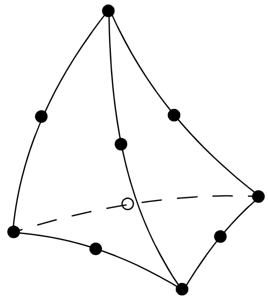
Tetrahedron (4 surfaces; 6 edges; 4 vertices):

(shape fcn. based on Pascal's pyramid => ***complete polynomial***)

Linear
(4 nodes)



Quadratic
(10 nodes)

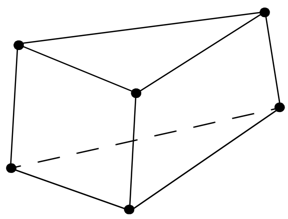


Basis functions: $\{1, \xi, \eta, \zeta\}$ $\{1, \xi, \eta, \zeta, \xi^2, \xi\eta, \xi\zeta, \eta^2, \eta\zeta, \zeta^2\}$

Prism (5 surfaces; 9 edges; 6 vertices):

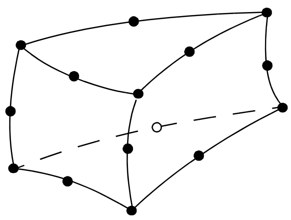
(shape fcn. based on an "extruded" triangle)

"Linear"
(6 nodes)



$\{1, \xi, \eta, \zeta, \underbrace{\xi\zeta, \eta\zeta}\}$

"quadratic"
(15 nodes)



$\{1, \xi, \eta, \zeta, \xi^2, \xi\eta, \xi\zeta, \eta^2, \eta\zeta, \zeta^2,$

$\underbrace{\xi^2\zeta, \xi\zeta^2, \eta^2\zeta, \eta\zeta^2, \xi\eta\zeta}\}$

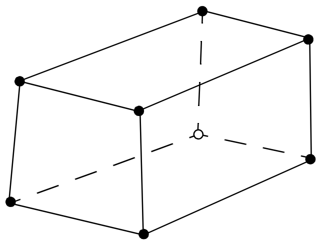
"extra terms to the complete polynomials"

cont. Examples of element types in 3D

<u>**Hexahedron** (6 surfaces; 12 edges; 8 vertices):</u>

(shape fcn. based on ***Lagrange interpolants***)

"Tri-linear"
(8 nodes)



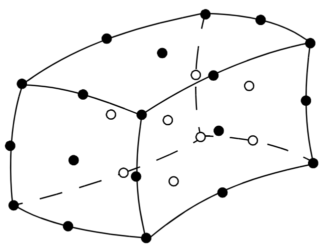
basis fcn.

$$\{(1 \quad \xi)(1 \quad \eta)(1 \quad \zeta)\} =$$

$$= \{1, \xi, \eta, \zeta, \xi\eta, \xi\zeta, \eta\zeta, \xi\eta\zeta\}$$

"Tri-quadratic"

"Full Lagrange"
(27 nodes)

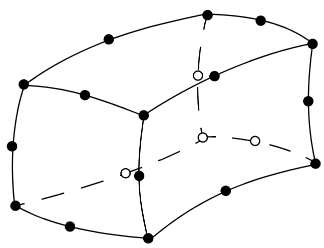


basis fcn.

$$\{(1 \ \xi \ \xi^2)(1 \ \eta \ \eta^2)(1 \ \zeta \ \zeta^2)\} =$$

$$= \{1, \xi, \eta, \zeta, \ldots, \xi^2\eta^2\zeta^2\} \Rightarrow 27 \text{ terms!}$$

"Serendipity"
(20 nodes)

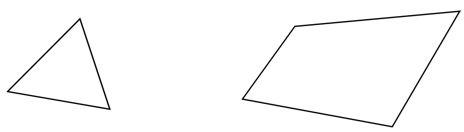


Surface & internal nodes removed! (in total 7)

Automatic meshing

<u>2*D*:</u>

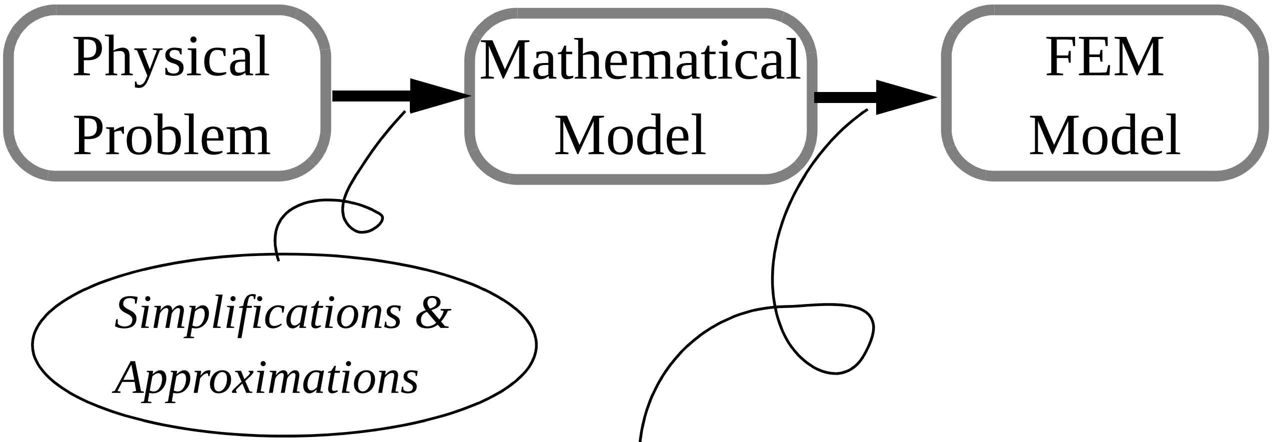
- Robust algorithms exist for arbitrary 2D domains for both 3- and 4-sided elements



<u>3*D*:</u>

- Robust algorithms for arbitrary volumes only exists for ***tetrahedron elements*** (10 nodes element, Note, the 4-node is never used!)
- Robust algorithms for ***hexahedron elements*** only exists for an "extruded" and "swept" geometry, where a mesh is generated with a plane surface as a starting point

6. Convergence & Sources of error in FEM



Sources of error

1. ***Discretization error*****:** incomplete representation of the geometry

2. ***Numerical error*****:** integration by Gauss quadrature; machine error (computer) round off/truncation depends on the conditions number of the system matrix **K**

3. ***Approximation error*****:** depends on interpolation of the primary variable

$$u \approx \tilde{u} = \sum_{e=1}^{N_e} \sum_{i=1}^{n_d} N_i u_i$$

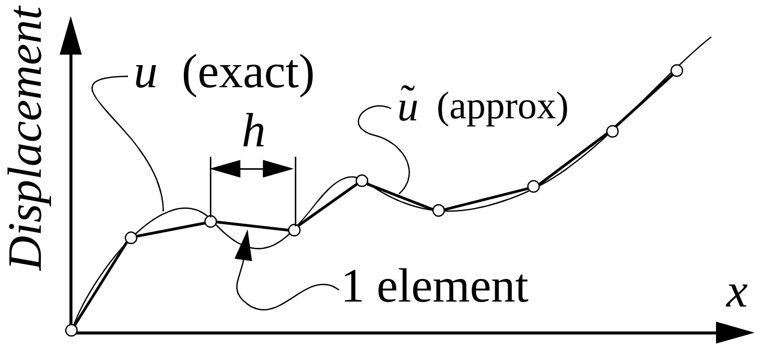
<u>Computational time:</u>
(CPU-time)

$$\boxed{t_{\mathrm{CPU}} \propto n_{\mathrm{Tot.DOF}}^{\alpha}}$$

$n_{\mathrm{Tot.DOF}}$ = total number of D.O.F. (number of equations)

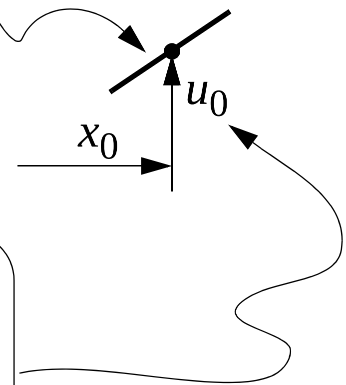
α is a constant in the interval 2 to 3, which depends on the type of equation solver and the type of system matrix (band width)

Convergence, 1D-example:



Features of the exact solution

$$u = u_0 + u'(x - x_0)$$

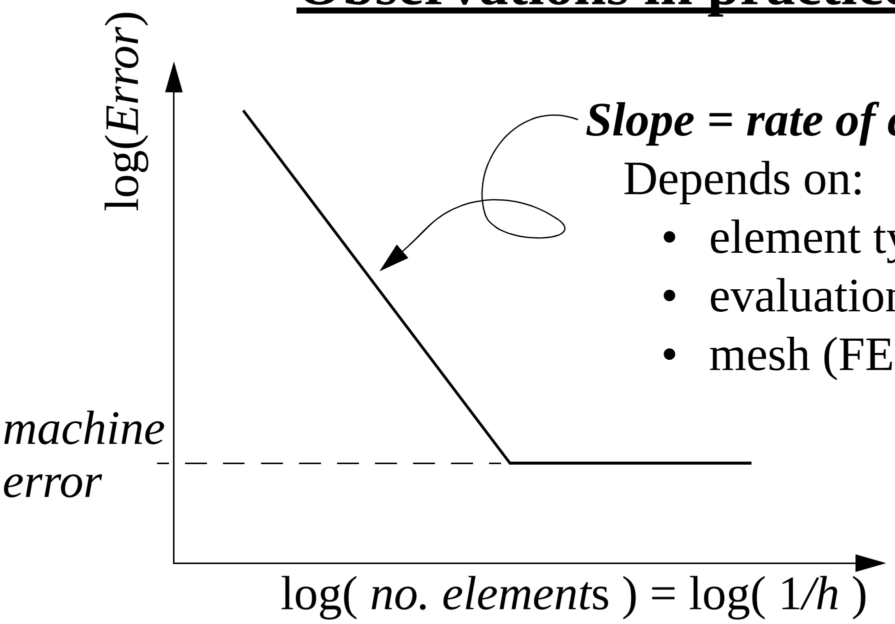


> *"The exact solution can in an infinitesimal region around a point x_0, be described by a constant plus a constant gradient"*

The conditions for convergence, such that $\tilde{u} \to u$ when $h \to 0$, require that the approximate displacement interpolation, $\tilde{u}$, must:

1. be able to describe
   (i) an arbitrary rigid body motion ($\sum N_i = 1$)
   (ii) a state of constant strain ($\mathrm{d}\tilde{u} / \mathrm{d}x = \text{constant}$)
2. be continuos across a element boundaries (compatibility)

<u>**Observations in practice**</u>



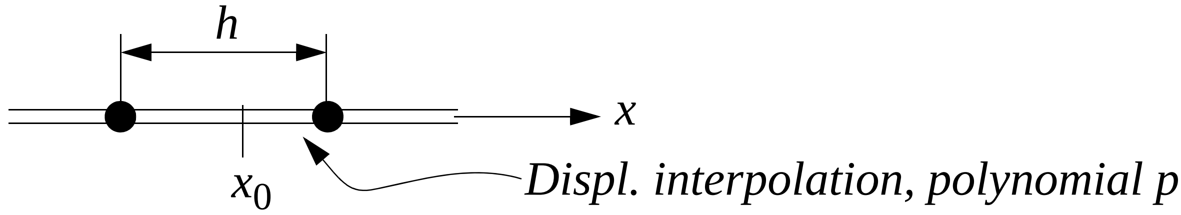
Slope = rate of convergence

Depends on:

- element type
- evaluation point
- mesh (FEM discretization)

Approximation error, 1*D*-example:

Study the solution in an 1*D* truss element:



Expansion of exact and approximate solution around x_0:

Exact: $u(x) = c_0 + c_1(x - x_0) + \ldots + c_p(x - x_0)^p + c_{p+1}(x - x_0)^{p+1} + \ldots$

Approx.: $\tilde{u}(x) = \tilde{c}_0 + \tilde{c}_1(x - x_0) + \ldots + \tilde{c}_p(x - x_0)^p$

The approximate solution will reproduce the exact solution up to polynomial degree p. The rest term, i.e. the error in the element will be of order $O((x - x_0)^{p+1})$, hence

$$Error = |\tilde{u} - u| \approx C_{p+1}(x - x_0)^{p+1} + \ldots$$

Since, maximum of (x-x_0) can be equal to h in an element, we obtain

$$Error = |\tilde{u} - u| \approx Ch^{p+1}$$

Logarithm

$$\Rightarrow \log(Error) = \log C + (p+1)\log h$$

$$= \log C - (p+1)\log\frac{1}{h}$$

$$\propto \log C - (p+1)\log(\textit{No. of elements})$$

Note!
No. of elements
is proportional
to 1 / h

Stress: $\sigma = E\varepsilon = E\dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\Rightarrow |\tilde{\sigma} - \sigma| = E\frac{\mathrm{d}}{\mathrm{d}x}|\tilde{u} - u| \approx Ch^p$$

Lower rate of convergence!

Optimal points for evaluation of results in an element

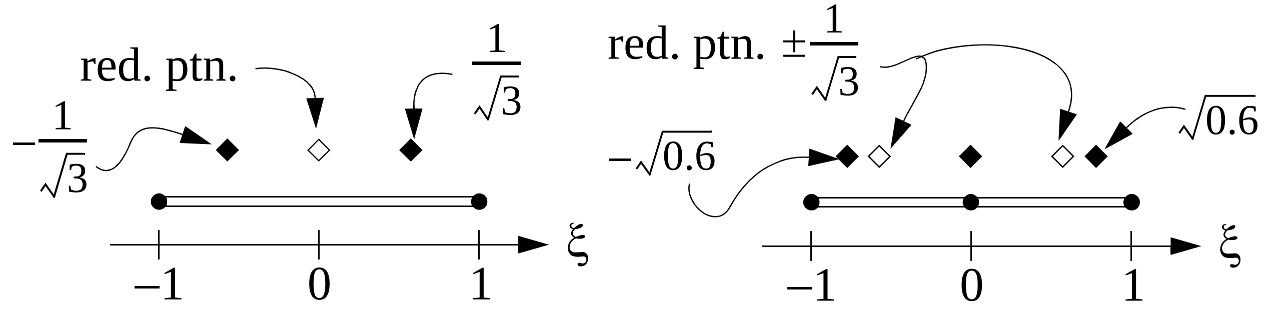
=> Optimal points often coincides with the integration points of "reduced integration"

Rep. Numerical integration with Gauss quadrature in 1*D*:

$$I = \int_{x_1}^{x_2} f(x)\mathrm{d}x = \int_{-1}^{1} f(\xi)\frac{\mathrm{d}x}{\mathrm{d}\xi}d\xi = \int_{-1}^{1} F(\xi)d\xi \approx \sum_{i=1}^{m} F(\xi_i)w_i$$

"integration point"

Element type	Number of integration points / element Full ◆	Reduced ◇
linear	2	1
quadratic	3	2



Linear   Quadratic

Methods for increased accuracy in a FEM analysis:

h–method: increase number of elements (i.e. decrease *h*)

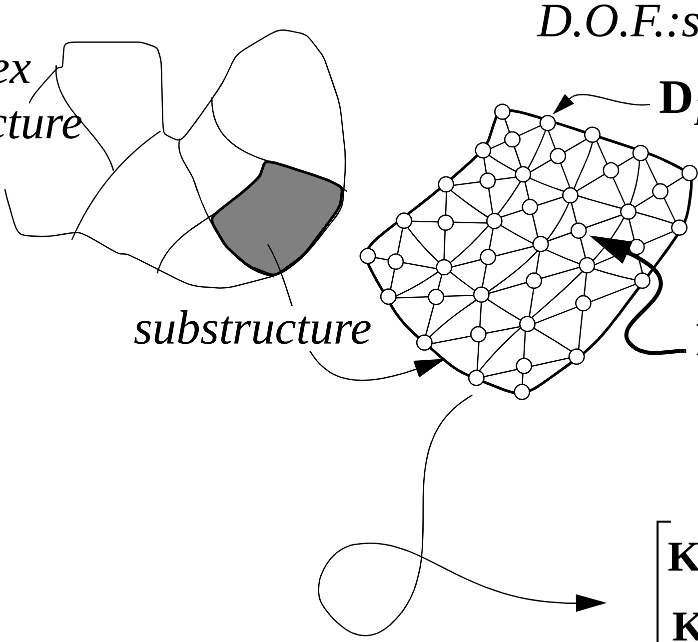
p–method: increase the polynomial degree in the interpolation

hp–method: combination of the h– & p–methods

r-method: use the existing elements in an optimized way, i.e. use biased meshes

7. Static condensation & Substructures

Method for analysis of very large complex systems, where the structure is divided into a smaller substructures. Each substructure is characterised by ***interior degrees of freedom (i)***, and ***degrees of freedom located on its boundary (b)***.



$$\begin{bmatrix} \mathbf{K}_{BB} & \mathbf{K}_{BI} \\ \mathbf{K}_{IB} & \mathbf{K}_{II} \end{bmatrix} \begin{bmatrix} \mathbf{D}_B \\ \mathbf{D}_I \end{bmatrix} = \begin{bmatrix} \mathbf{F}_B \\ \mathbf{F}_I \end{bmatrix}$$

Three step procedure:

(*i*) ***Eliminate*** the ***interior degrees of freedom*** in each substructure. Each substructure can then be described by only $\boldsymbol{m}$ D.O.F.:s instead of the $\boldsymbol{m} + \boldsymbol{n}$ D.O.F.:s for the whole substructure. This step is called *static condensation.*

(*ii*) ***Assemble the substructures*** and analyse the whole system. *Note that, the number of equations that must be solved simultaneously are now drastically reduced!*

(*iii*) Evaluate the results in each substructure (post-processing).

substructure cont.

Step (i) Elimination of interior D.O.F.:s in a substructure

Divide the equation system in boundary (B) and interior (I) degrees of freedom

$$\begin{bmatrix} \mathbf{K}_{BB} & \mathbf{K}_{BI} \\ \mathbf{K}_{IB} & \mathbf{K}_{II} \end{bmatrix} \begin{bmatrix} \mathbf{D}_R \\ \mathbf{D}_I \end{bmatrix} = \begin{bmatrix} \mathbf{F}_B \\ \mathbf{F}_I \end{bmatrix} \qquad \text{(Note that } \mathbf{K}_{BI} = \mathbf{K}_{IB}^T\text{)}$$

with dimensions: $\mathbf{K}_{BB}$ $m \times m$; $\mathbf{K}_{II}$ $n \times n$; $\mathbf{K}_{IB}$ $n \times m$; $\mathbf{K}_{BI}$ $m \times n$.

Eliminate $\mathbf{D}_I$ using the lower equation $\mathbf{K}_{IB}\mathbf{D}_B + \mathbf{K}_{II}\mathbf{D}_I = \mathbf{F}_I$

$$\Rightarrow \mathbf{D}_I = \mathbf{K}_{II}^{-1}\mathbf{F}_I - \mathbf{K}_{II}^{-1}\mathbf{K}_{IB}\mathbf{D}_B,$$

which inserted into the upper equation $\mathbf{K}_{BB}\mathbf{D}_B + \mathbf{K}_{BI}\mathbf{D}_I = \mathbf{F}_B$ gives

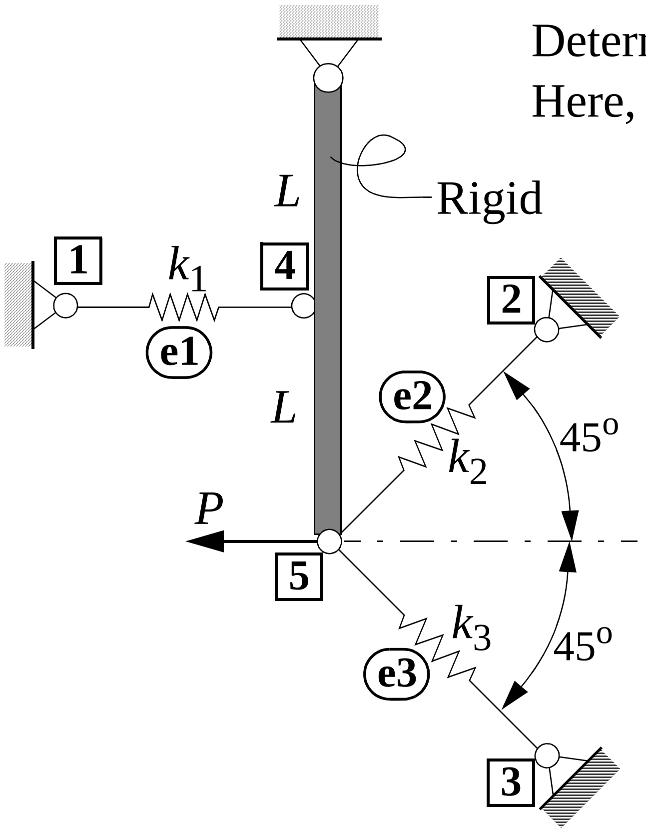
$$\Rightarrow [\mathbf{K}_{BB} - \mathbf{K}_{BI}\mathbf{K}_{II}^{-1}\mathbf{K}_{IB}]\mathbf{D}_B = \mathbf{F}_B - \mathbf{K}_{BI}\mathbf{K}_{II}^{-1}\mathbf{F}_I$$

Thus, the mechanical response of the substructure can be modelled by the reduced equation system

$$\mathbf{K}_{red}\mathbf{D}_B = \mathbf{F}_{red},$$

where $\mathbf{K}_{red} = \mathbf{K}_{BB} - \mathbf{K}_{BI}\mathbf{K}_{II}^{-1}\mathbf{K}_{IB}$ and $\mathbf{F}_{red} = \mathbf{F}_B - \mathbf{K}_{BI}\mathbf{K}_{II}^{-1}\mathbf{F}_I$.

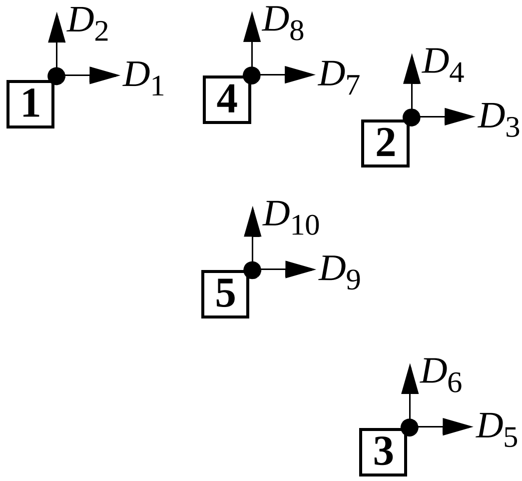
8. Constraint Equations—An example



Determine the displace. in direction of P

Here, $k_1 = k$, $k_2 = k_3 = 2k$

Numbering of D.O.F.



Boundary conditions:

$D_1 = ... = D_6 = 0$, $D_8 = D_{10} = 0$

Note! D_7 and D_9 in not independent!

Element stiffness matrices:

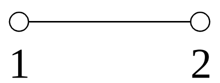
"Direction cosines"

$$\mathbf{k}_e = k_i \begin{bmatrix} \mathbf{a} & -\mathbf{a} \\ -\mathbf{a} & \mathbf{a} \end{bmatrix}; \quad \mathbf{a} = \begin{bmatrix} l_{12}^2 & l_{12}m_{12} \\ l_{12}m_{12} & m_{12}^2 \end{bmatrix}$$

$$l_{12} = \frac{x_2 - x_1}{l_i}, \quad m_{12} = \frac{y_2 - y_1}{l_i}$$

element length

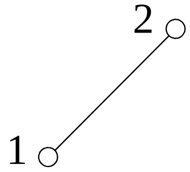
<u>Elem. 1:</u>



$$l_{12} = 1$$
$$m_{12} = 0$$

$$\Rightarrow k_1 = k, \quad \mathbf{a} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

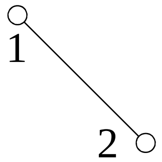
<u>Elem. 2:</u>



$$l_{12} = 1/\sqrt{2}$$
$$m_{12} = 1/\sqrt{2}$$

$$\Rightarrow k_2 = 2k, \quad \mathbf{a} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

<u>Elem. 3:</u>



$$l_{12} = 1/\sqrt{2}$$
$$m_{12} = -1/\sqrt{2}$$

$$\Rightarrow k_3 = 2k, \quad \mathbf{a} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembly of element stiffness matrices:

element 1 — marked by circle

element 2 & 3 — unmarked

$$
\mathbf{K}_{tot} = \mathbf{K}_{e1} + \mathbf{K}_{e2} + \mathbf{K}_{e3} = k
\begin{array}{c c}
\left[\begin{array}{cc|cc|cc|cc|cc}
1 & 0 & & & & & -1 & 0 & & \\
0 & 0 & \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & 0 & 0 & \multicolumn{2}{c}{\mathbf{0}} \\
\hline
 & & 1 & 1 & & & & & -1 & -1 \\
\multicolumn{2}{c|}{\mathbf{0}} & 1 & 1 & \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & -1 & -1 \\
\hline
 & & & & 1 & -1 & & & -1 & 1 \\
\multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & -1 & 1 & \multicolumn{2}{c|}{\mathbf{0}} & 1 & -1 \\
\hline
-1 & 0 & & & & & 1 & 0 & & \\
0 & 0 & \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & 0 & 0 & \multicolumn{2}{c}{\mathbf{0}} \\
\hline
 & & -1 & -1 & -1 & 1 & & & 2 & 0 \\
\multicolumn{2}{c|}{\mathbf{0}} & -1 & -1 & 1 & -1 & \multicolumn{2}{c|}{\mathbf{0}} & 0 & 2
\end{array}\right]
&
\begin{array}{ll}
1 & \\
2 & \text{nod 1} \\
3 & \\
4 & \text{nod 2} \\
5 & \\
6 & \text{nod 3} \\
7 & \\
8 & \text{nod 4} \\
9 & \\
10 & \text{nod 5}
\end{array}
\\
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
& \text{Eqn.no.}
\end{array}
$$

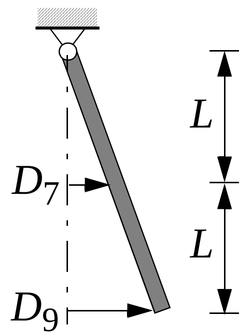
Equations 7 & 9 (other $D_i = 0$, i.e. known):

$$
\begin{bmatrix} k & 0 \\ 0 & 2k \end{bmatrix} \begin{bmatrix} D_7 \\ D_9 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix}
$$

But, the solution to the Eq. system is not the solution to the problem!

Since, D_7 and D_9 is linear dependent:

$$\Rightarrow D_7 = \frac{1}{2} D_9$$



Methods for solving problems with constraints:

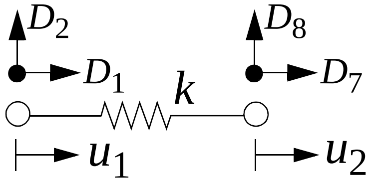
A. Constraint introduced on element level (elimination of D.O.F.), affect the transformation $\mathbf{k}_e \rightarrow \mathbf{K}_e$.

B. Lagrange multiplier method

C. Penalty method

A. Constraint introduced on the element level:

Element 1:



Constraint: $D_7 = \frac{1}{2}D_9$

$D_8 = D_{10}$

Relation local/global displacements:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_7 \\ D_8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} D_1 \\ D_2 \\ D_9 \\ D_{10} \end{bmatrix}$$

New transformation matrix

Derive a new global element stiffness matrix by use of the standard rule transformation, i.e.

$$\mathbf{K}_{e1} = \mathbf{T}^T\mathbf{k}_e\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} k & 0 & -k/2 & 0 \\ 0 & 0 & 0 & 0 \\ -k/2 & 0 & k/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The degrees of freedom D7 & D9 are now eliminated from the Eq. system, and the global stiffness matrix becomes

element 1 — marked by dashed circle
element 2 & 3 — unmarked
element 1, 2 & 3 — marked by circle

$$\mathbf{K} = \mathbf{K}_{e1} + \mathbf{K}_{e2} + \mathbf{K}_{e3} = k \begin{array}{c} \left[\begin{array}{cc:cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ \hdashline 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ \hdashline -1/2 & 0 & -1 & -1 & -1 & 1 & 9/4 & 0 \\ 0 & 0 & -1 & -1 & 1 & -1 & 0 & 2 \end{array}\right] \\ \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 9 & 10 \end{array} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 9 \\ 10 \\ \text{Eqn. no.} \end{array}$$

Equation 9 (other $D_i = 0$, i.e. known):

$$\begin{bmatrix} \frac{9k}{4} \end{bmatrix} \begin{bmatrix} D_9 \end{bmatrix} = \begin{bmatrix} -P \end{bmatrix} \quad \Rightarrow \quad D_9 = -\frac{4P}{9k}$$

Constraint equation—General form

(Se Chap. 11.11 in the textbook, where the terminology "MPC-Equations", where "MPC" stands for Multi-Point-Constraint")

Constraint equations can be formulated on the general form:

$$\underset{m\times n}{\mathbf{C}}\,\underset{n\times 1}{\mathbf{D}} - \underset{m\times 1}{\mathbf{Q}} = \mathbf{0}$$

n = number of D.O.F.

m = number of constraints

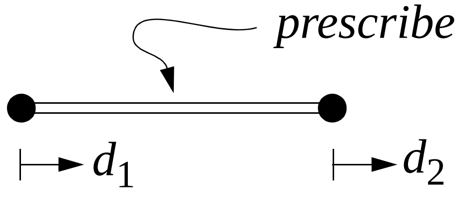
In the present case (Eq. (7) & (9)) above, yields that $n = 2$ & $m = 1$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -1/2 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} D_7 \\ D_9 \end{bmatrix}}_{\mathbf{D}} - \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{Q}} = \begin{bmatrix} 0 \end{bmatrix}$$

In a typical case, **Q** *represents a prescribed displacement!*

E.g. the extension of an element can be described by **Q** *as*

prescribe an extension $\delta = d_2 - d_1$



Can be formulated by use a constraint according to:

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} - \begin{bmatrix} \delta \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

B. Lagrange multiplier method:

(See Chap. 11.11 in textbook)

The ***potential energy*** $U(\mathbf{D})$ of a system is minimum at a point of ***stable equilibrium***, i.e. $\delta U(\mathbf{D}) = 0$.

Here, a unique equilibrium solution can be derived by minimization of *the potential* $U(\mathbf{D})$ under the *constraint*: $\mathbf{CD} - \mathbf{Q} = \mathbf{0}$ ("optimization problem"). This can be accomplished by minimizing the *modified potential*:

$$U_L = \underbrace{\frac{1}{2}\mathbf{D}^T\mathbf{K}\mathbf{D}}_{\text{Elastic strain energy}} - \underbrace{\mathbf{D}^T\mathbf{F}}_{\text{External force potential}} + \boldsymbol{\lambda}^T\underbrace{(\mathbf{C}\mathbf{D} - \mathbf{Q})}_{\text{Constraint}}$$

Lagrange-multipliers

Minimization w.r.t. $\mathbf{D}$ and λ gives

$$\delta U = \delta\mathbf{D}^T(\mathbf{K}\mathbf{D} - \mathbf{F}) + \delta\boldsymbol{\lambda}^T(\mathbf{C}\mathbf{D} - \mathbf{Q}) + \overbrace{\boldsymbol{\lambda}^T\mathbf{C}\delta\mathbf{D}}^{\delta\mathbf{D}^T\mathbf{C}^T\boldsymbol{\lambda}} = 0$$

Minumum is a stationary solution, i.e. $\delta U = 0$ is valid for arbitrary $\delta\mathbf{D}$ and $\delta\lambda$, which gives the equation system

$$\begin{bmatrix} \mathbf{K} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{Q} \end{bmatrix}$$

In the present example the equation system & solution become

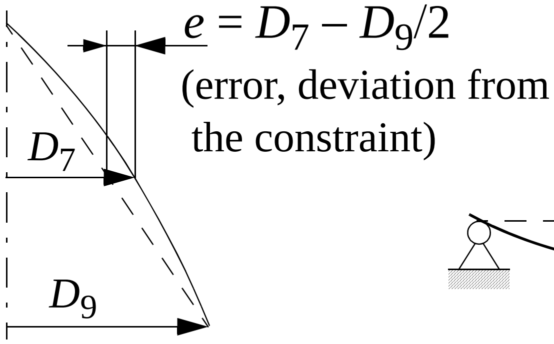
$$\begin{bmatrix} k & 0 & 1 \\ 0 & 2k & -1/2 \\ 1 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} D_7 \\ D_9 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -P \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} D_7 \\ D_9 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2P/(9k) \\ -4P/(9k) \\ 2P/9 \end{bmatrix}$$

(The method is used in many FEM codes, e.g. ABAQUS, ANSYS)

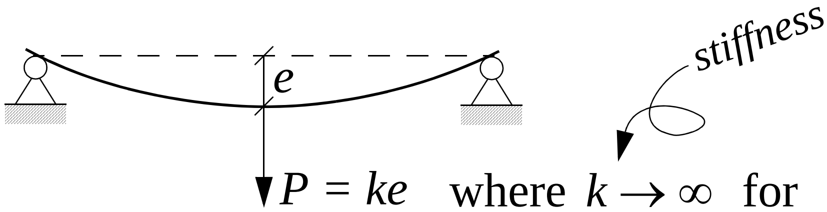
C. Penalty method:

(See Chap. 11.11 in text book)

The constraint in the example is based on the assumption of a rigid beam. If the beam is treated as not being rigid, it will have certain flexibility and the constraint will not be completely satisfied, which give rise to an error e according to



e describes the deformation of the beam, which can be expressed by a simple model according to



where $k \to \infty$ for a *rigid beam*

If $k < \infty$, **elastic energy** will be stored in the beam according to

$$W_{\text{balk}} = \frac{1}{2}eP = \frac{1}{2}eke,$$

which will increase the *potential energy* of the system.

In general, the error in satisfying the constraint can be expressed as

$$\mathbf{e} = \mathbf{CD} - \mathbf{Q},$$

where **e** is a vector of dimension equal to the number constraint Eqs. The error **e** will contribute to the potential energy of the system as

$$U = \frac{1}{2}\mathbf{D}^T\mathbf{KD} - \mathbf{D}^T\mathbf{F} + \frac{1}{2}\mathbf{e}^T\mathbf{k}^p\mathbf{e},$$

dimension energy

Here, $\mathbf{k}^p$ is a diagonal matrix containing stiffness terms, k_i^p, according to the example above, which here is called “penalty”-numbers.

cont. C. Penalty method

The best solution is obtained by *minimizing the potential energy* of the system w.r.t. **D**, which here gives

$$\delta U = \delta\mathbf{D}^T(\mathbf{KD}-\mathbf{F}) + \overbrace{\delta\mathbf{e}^T}^{\delta\mathbf{D}^T\mathbf{C}^T}(\mathbf{k}^p(\mathbf{CD}-\mathbf{Q})) = 0$$

The stationary solution must be valid for arbitrary $\delta\mathbf{D}$, which gives the equation system:

$$[\mathbf{K} + \underbrace{\mathbf{C}^T\mathbf{k}^p\mathbf{C}}_{\textit{Penalty matrix}}]\mathbf{D} = \mathbf{F} + \mathbf{C}^T\mathbf{k}^p\mathbf{Q}$$

The "Penalty numbers" are chosen by the analyst, and can be chosen according to

$$k_i^p = [10^4 \text{ till } 10^8] \times [\text{maximal diagonal element in } \mathbf{K}]$$

If k_i^p is chosen to big, the system matrix becomes ill-conditioned!

Applied to the example above gives

$$\mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & 2k \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0 \\ -P \end{bmatrix}, \mathbf{C}^T = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 0 \end{bmatrix}$$

Explore different penalty numbers in the diagonal matrix α

$$\mathbf{k}^p = \begin{bmatrix} 10^2 k \end{bmatrix} \Rightarrow \mathbf{C}^T\mathbf{k}^p\mathbf{C} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}\begin{bmatrix} 10^2 k \end{bmatrix}\begin{bmatrix} 1 & -1/2 \end{bmatrix} = 10^2 k\begin{bmatrix} 1 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

Eqs. syst.:

$$10^2 k\begin{bmatrix} 1.01 & -0.5 \\ -0.5 & 0.27 \end{bmatrix}\begin{bmatrix} D_7 \\ D_9 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix} \Rightarrow \begin{bmatrix} D_7 \\ D_9 \end{bmatrix} = -\frac{P}{k}\begin{bmatrix} 0.2203 \\ 0.4449 \end{bmatrix}$$

Relative error $< 10^{-2}$

cont. C. Penalty method

$$\mathbf{k}^p = \begin{bmatrix} 10^4 k \end{bmatrix} \Rightarrow \mathbf{C}^T\mathbf{k}^p\mathbf{C} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \begin{bmatrix} 10^4 k \end{bmatrix} \begin{bmatrix} 1 & -1/2 \end{bmatrix} = 10^4 k \begin{bmatrix} 1 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

Eqs. syst.:

$$10^4 k \begin{bmatrix} 1.0001 & -0.5 \\ -0.5 & 0.2502 \end{bmatrix} \begin{bmatrix} D_7 \\ D_9 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} D_7 \\ D_9 \end{bmatrix} = -\frac{P}{k} \begin{bmatrix} 0.222202 \\ 0.444449 \end{bmatrix}$$

Relative error $< 10^{-4}$

Method	*advantage*	*disadvantage*
Lagrange	exact	adds extra D.O.F.
Penalty(*)	no new D.O.F.	not exact

(*) The method is often used in contact analysis, where constraint equations can be formulated as an inequality.

Lectures 16 & 17:

1. Heat conduction—fundamental relations (1*D*/2*D*)

2. FEM-Eq. for heat conduction in 1*D* (LQ, chap. 12)

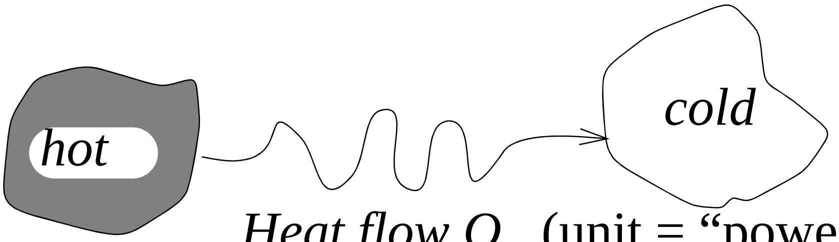
3. Example: Thermal FEM analysis in 1*D*

4. FEM-Eq. for thermo-elastic materials

5. Example: Mechanical FEM-analysis with temperature load

6. FEM-Eq. for heat conduction in 2*D*

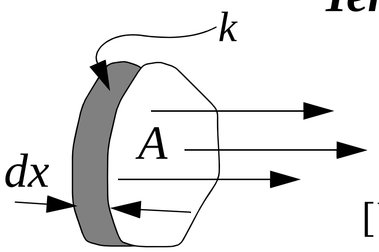
Heat Conduction



Heat flow Q (unit = "power" [W])

Fourier's law:
("constitutive law")

*Medium with cross sectional area **A***
*coefficient of thermal conductivity **k***



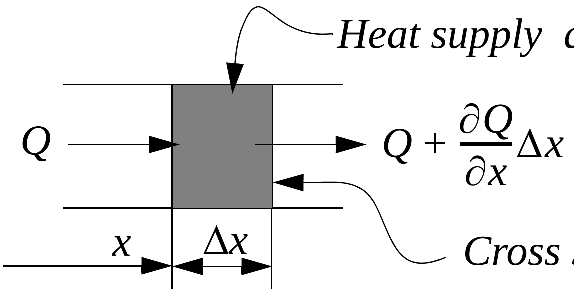
Temperature

$$Q = -\frac{\partial T}{\partial x} \cdot A \cdot k$$

$$[\mathrm{W}] \qquad \left[\frac{°\mathrm{C}}{\mathrm{m}}\right] \; [\mathrm{m}^2] \; \left[\frac{\mathrm{W}}{\mathrm{m}\,°\mathrm{C}}\right]$$

Energy balance / unit time (1D):

Heat supply q $[\mathrm{W/m}^3]$



Cross sectional area A

[heat flow in – heat flow out] + [heat generated]

= [heat change in the element]

$$\left[Q - \left(Q + \frac{\partial Q}{\partial x}\Delta x\right)\right] + [q \cdot \overbrace{A\Delta x}^{\textit{volume element}}] = \left[\frac{\partial T}{\partial t} \cdot c \cdot \overbrace{A\Delta x\rho}^{\textit{mass element}}\right]$$

Specific heat $\left[\frac{\mathrm{J}}{\mathrm{kg}\,°\mathrm{C}}\right]$

Let $\Delta x \to 0$: $\Rightarrow$ $-\frac{\partial Q}{\partial x} + qA = cA\rho\frac{\partial T}{\partial t}$

With Fourier's law we obtain: $\frac{\partial}{\partial x}\left(Ak\frac{\partial T}{\partial x}\right) + qA = cA\rho\frac{\partial T}{\partial t}$

Steady state conditions if $\partial T/\partial t = 0$

=> *special case of great technical importance, treated here!*

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + qA = 0$$

Boundary cond.

Boundary with cross sectional area A

Ambient temperature of surrounding medium T_∞

(i) prescribed temperature T

(ii) prescribed heat flow $Q = -kA\, dT/dx$

special case $Q = 0$ (insulated surface, no heat exchange with the surrounding medium)

(iii) Special boundary cond.:

Convection coefficient $\left[\frac{W}{m^2\,°C}\right]$

Convection: $Q = h(T - T_\infty)A$

Radiation: $Q = \varepsilon\sigma(T^4 - T_\infty^4)A$ (not treated here!)

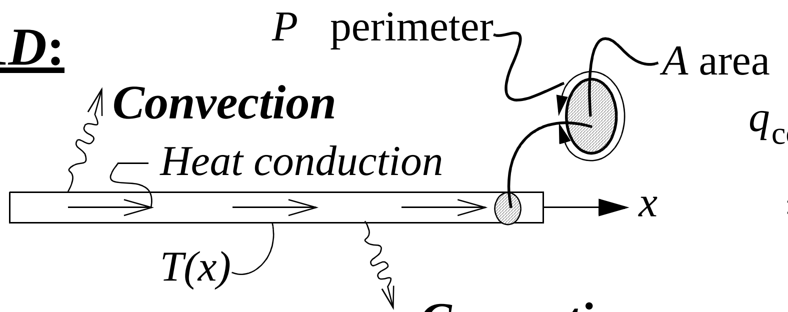
Emissivity of the surface

Stefan-Bolzmann constant

Convection of heat through surfaces between boundaries

"acts as a negative heat supply in 1D & 2D analyses"

1D:

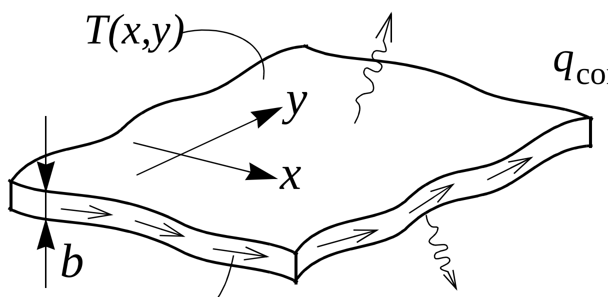


$$q_{conv} A \cancel{\Delta x} = hP\cancel{\Delta x}(T - T_\infty)$$

$$\Rightarrow q_{conv} A = hP(T - T_\infty)$$

unit: [W/m]

2D:



$$q_{conv} b \cancel{\Delta x}\cancel{\Delta y} = 2h\cancel{\Delta x}\cancel{\Delta y}(T - T_\infty)$$

$$\Rightarrow q_{conv} b = 2h(T - T_\infty)$$

unit: $[W/m^2]$

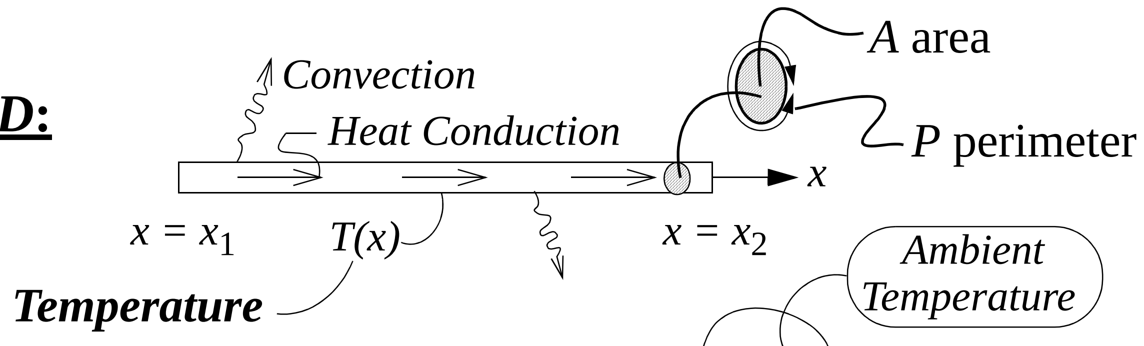
Heat conduction

Convection (both through the top and the bottom surface, respectively)

Heat Transfer—Summary of 1*D* & 2*D* models

Temperature under steady state conditions $\left(\frac{\partial T}{\partial t} = 0\right)$

1*D*:



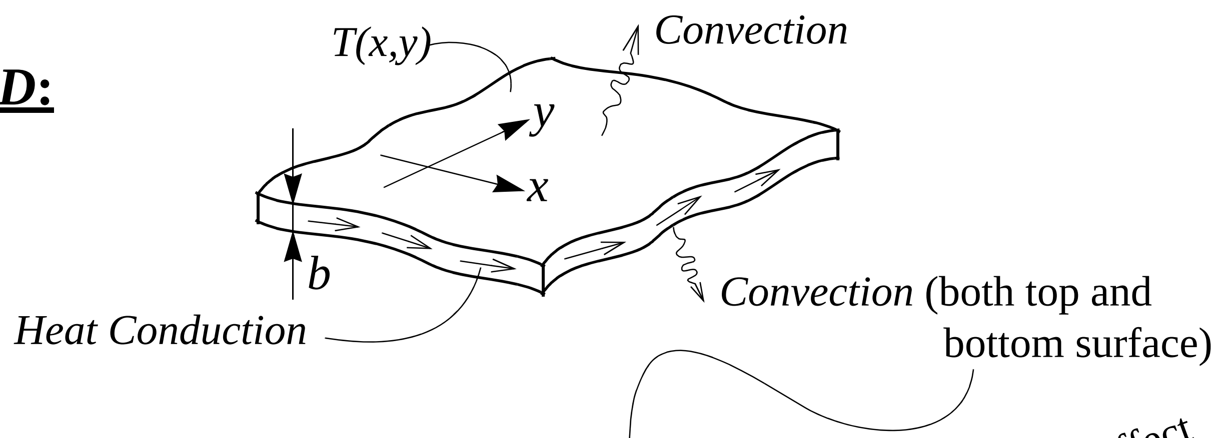
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(kA\frac{dT}{\mathrm{d}x}\right) + qA - hP(T - T_\infty) = 0 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}}\right] \qquad \frac{\text{Effect}}{\text{Unit length}}$$

Thermal Conductivity $\left[\frac{\mathrm{W}}{\mathrm{m}\,^\circ\mathrm{C}}\right]$

Heat supply $\left[\frac{\mathrm{W}}{\mathrm{m}^3}\right]$

Convection coefficient $\left[\frac{\mathrm{W}}{\mathrm{m}^2\,^\circ\mathrm{C}}\right]$

2*D*:



$$\frac{\partial}{\partial x}\left(k_x b\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y b\frac{\partial T}{\partial y}\right) + qb - 2h(T - T_\infty) = 0 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right] \qquad \frac{\text{Effect}}{\text{Unit area}}$$

Boundary Conditions: prescribed temperature T

or heat flow $Q = -kA\frac{\mathrm{d}T}{\mathrm{d}x}$ [W]

Special B.C: insolated surface $Q = 0$

convection from end surface $Q = hA(T - T_\infty)$

FEM-Equations (1*D*)—by use of *weak form*

1. Weighted residual (differential equation multiplied by arbitrary weight function *v(x)* and integrate)

$$\int_{x_1}^{x_2} v(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right) + qA - hP(T - T_\infty)\right)\mathrm{d}x = 0$$

2. Integration by parts (1st term)

$$\int_{x_1}^{x_2} v\frac{\mathrm{d}}{\mathrm{d}x}\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x = \left[vkA\frac{\mathrm{d}T}{\mathrm{d}x}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}v}{\mathrm{d}x}\, kA\, \frac{\mathrm{d}T}{\mathrm{d}x}\, \mathrm{d}x$$

gives the weak form of the heat transfer problem in 1*D*:

$$\int_{x_1}^{x_2} \frac{\mathrm{d}v}{\mathrm{d}x}\, kA\, \frac{\mathrm{d}T}{\mathrm{d}x}\, \mathrm{d}x + \int_{x_1}^{x_2} vhPT\mathrm{d}x = \left[vkA\frac{\mathrm{d}T}{\mathrm{d}x}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} vqA\mathrm{d}x + \int_{x_1}^{x_2} vhPT_\infty\mathrm{d}x$$

Divide into elements and formulate an approximate interpolation of the temperature by use of standard shape functions. Use the same shape functions to express the weight functions (Galerkin):

Temperature: $T(x) = \mathbf{N}\mathbf{T}_e$ — *vector containing the node temperature of the element*

Weight function: $v(x) = \mathbf{N}\boldsymbol{\beta} = \boldsymbol{\beta}^T\mathbf{N}^T$ — *arbitrary vector*

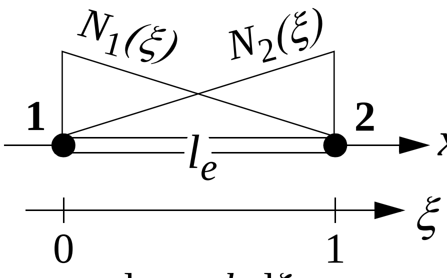
Derivatives: $\dfrac{\mathrm{d}T}{\mathrm{d}x} = \dfrac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\mathbf{T}_e = \mathbf{B}\mathbf{T}_e$ $\qquad \dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{\mathrm{d}\mathbf{N}^T}{\mathrm{d}x}\boldsymbol{\beta}^T = \mathbf{B}^T\boldsymbol{\beta}^T$

FEM-Equation for an element becomes: $\qquad kA\mathrm{d}T/\mathrm{d}x = -Q$

$$\Bigg[\underbrace{\int_{l_e}\mathbf{B}^T kA\mathbf{B}\mathrm{d}x}_{\mathbf{K}_{hc}} + \underbrace{\int_{l_e}\mathbf{N}^T hP\mathbf{N}\mathrm{d}x}_{\mathbf{K}_c}\Bigg]\mathbf{T}_e = \underbrace{[\mathbf{N}^T(-Q)]_{x_1}^{x_2}}_{} + \underbrace{\int_{l_e}\mathbf{N}^T qA\mathrm{d}x + \int_{l_e}\mathbf{N}^T hPT_\infty\mathrm{d}x}_{\mathbf{f}_b}$$

If a heat supply is prescribed at the node, or if element boundary = external boundary

Example: linear temperature interpolation (1*D*)



$\mathrm{d}x = l_e \mathrm{d}\xi$

Thermal Conductivity — *Convection coeff.* — *area* — *perimeter*

Assume that k, h, A and P are constant in the element

Temperature:

$$T = \underbrace{(1-\xi)}_{N_1} T_1 + \underbrace{\xi}_{N_2} T_2 = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \mathbf{N}\mathbf{T}_e$$

Temperature gradient:

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\mathbf{T}_e = \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x}\mathbf{T}_e = \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix}\mathbf{T}_e = \mathbf{B}\mathbf{T}_e$$

Element matrices

<u>L.h.s:</u>

Heat Conduction

$$\mathbf{K}_{hc} = \int_0^1 \mathbf{B}^T kA\mathbf{B} l_e \mathrm{d}\xi = \frac{kA}{l_e}\int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}\mathrm{d}\xi = \frac{kA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Heat Convection

$$\mathbf{K}_{c} = \int_0^1 \mathbf{N}^T hP\mathbf{N} l_e \mathrm{d}\xi = hPl_e\int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}\begin{bmatrix} 1-\xi & \xi \end{bmatrix}\mathrm{d}\xi = \frac{hPl_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

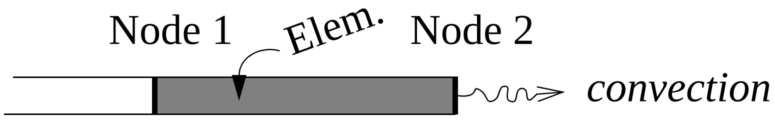
<u>R.h.s:</u>

$$\mathbf{f}_b = \int_0^1 \mathbf{N}^T qAl_e \mathrm{d}\xi + \int_0^1 \mathbf{N}^T hPT_\infty l_e \mathrm{d}\xi =$$

$$= Al_e\int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} q(\xi)\mathrm{d}\xi + hPT_\infty l_e \underbrace{\int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}\mathrm{d}\xi}_{= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}$$

cont. 1D–Example

R.h.s. cont.: *If e.g. node 2 is a node on the boundary and a convection B.C. is employed, then*



$$\overrightarrow{Q_1} \qquad \overleftarrow{Q_2} = -hA(T_2 - T_\infty)$$

Note that Q_1 is cancelled by $-Q_2$ in the left element and do not enter the r.h.s when all element contributions are assembled

$$\overrightarrow{-Q} = kA\frac{\mathrm{d}T}{\mathrm{d}x}$$

$$\Rightarrow [\mathbf{N}^T(-Q)]_0^1 = \begin{bmatrix} 0 \\ Q_2 \end{bmatrix} - \begin{bmatrix} -Q_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ -hA(T_2 - T_\infty) \end{bmatrix} =$$

$$= -\begin{bmatrix} 0 \\ hAT_2 \end{bmatrix} + \begin{bmatrix} Q_1 \\ hAT_\infty \end{bmatrix} = -\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}}_{\mathbf{K}_r}\underbrace{\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}}_{\mathbf{T}_e} + \underbrace{\begin{bmatrix} 0 \\ hAT_\infty \end{bmatrix}}_{\mathbf{f}_s} + \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}$$

Move to L.h.s.

Heat conduction

Convection from the "inside" the element

Convection at a node placed on the boundary

Totally $$\underbrace{[\mathbf{K}_{hc} + \mathbf{K}_c + \mathbf{K}_b]}_{\mathbf{K}_e}\mathbf{T}_e = \underbrace{\mathbf{f}_s + \mathbf{f}_b}_{\mathbf{f}_e}$$

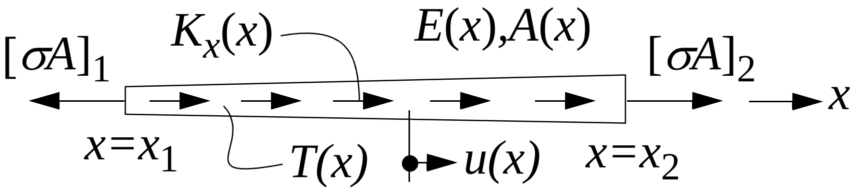
Thus, the FEM Eq. for one element becomes $$\mathbf{K}_e\mathbf{T}_e = \mathbf{f}_e$$

The total (global) equation system is obtained by assembly of all element matrices as

Number of elements

$$\mathbf{K} = \sum_{e=1}^{N_e}\mathbf{K}_e \qquad \mathbf{F} = \sum_{e=1}^{N_e}\mathbf{F}_e$$

FEM for Thermo-Elastic materials (1*D*)



<u>*Equilibrium:*</u>

$$\frac{\mathrm{d}(\sigma A)}{\mathrm{d}x} + K_x A = 0$$

Equilibrium Eq. inserted into ***Weak Form*** gives

$$\int_{x_1}^{x_2} \frac{\mathrm{d}v}{\mathrm{d}x}(\sigma A)\mathrm{d}x = [v(\sigma A)]_{x_1}^{x_2} + \int_{x_1}^{x_2} v K_x A \mathrm{d}x$$

Constitutive Equation

Thermal expansion coefficient [1/°C]

$\alpha \Delta T$

$$\varepsilon = \frac{\sigma}{E} + \alpha \Delta T \quad \Leftrightarrow \quad \sigma = E\varepsilon - E\varepsilon_0$$

Temperature change from reference temp.

Inserted into weak form with ε = d*u*/d*x* gives

New term!

$$\int_{x_1}^{x_2} \frac{\mathrm{d}v}{\mathrm{d}x} EA \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \underbrace{[v(\sigma A)]_{x_1}^{x_2}}_{\text{Boundary forces}} + \underbrace{\int_{x_1}^{x_2} v K_x A \mathrm{d}x}_{\text{Body force}} + \boxed{\underbrace{\int_{x_1}^{x_2} \frac{\mathrm{d}v}{\mathrm{d}x} EA\varepsilon_0 \mathrm{d}x}_{\text{Thermal load}}}$$

FEM Eq. for one element ($u = \mathbf{N}\mathbf{d}_e$ & $v = \mathbf{N}^T\boldsymbol{\beta}^T$)

Extra term in the force vector

$$\underbrace{\left[\int_{x_1}^{x_2} \mathbf{B}^T EA \mathbf{B} \mathrm{d}x\right]}_{\mathbf{k}_e} \mathbf{d}_e = \underbrace{[N^T(\sigma A)]_{x_1}^{x_2}}_{\mathbf{f}_s} + \underbrace{\int_{x_1}^{x_2} \mathbf{N}^T K_x A \mathrm{d}x}_{\mathbf{f}_b} + \boxed{\underbrace{\int_{x_1}^{x_2} \mathbf{B}^T EA\varepsilon_0 \mathrm{d}x}_{\mathbf{f}_T}}$$

Stress calculations in the “post processing” step:

*Use same interpolation as for the **mechanical strain***

$$\sigma = E\varepsilon - E\varepsilon_0 = E\frac{du}{dx} - E\varepsilon_0 = E\mathbf{B}\mathbf{d}_e - E\varepsilon_0$$

The 2*D* & 3*D* formulations are analogous with the 1*D* formulation

Computational steps: Thermo–Elastic Analysis

1. Discretization: divide the solid into elements. It is convenient to use the same mesh in both the thermal and the mechanical analysis.

2. Carry out the ***thermal analysis*** (solve the heat transfer problem). The result, i.e. the temperature distribution in the solid, is presented as **temperatures at the nodes**.

3. Carry out the ***mechanical analysis*** (stress analysis). It is convenient to use the same interpolation (shape fcn.) for the displacement as used for the temperature.

same shape fcn. as in the interpolation of the displacement

<u>1*D* example:</u>

Result from thermal analysis $\Rightarrow \Delta T(x) = \mathbf{N}\,\underbrace{\Delta\mathbf{T}_e}_{\text{Temperature change in the nodes}}$

Thermal load

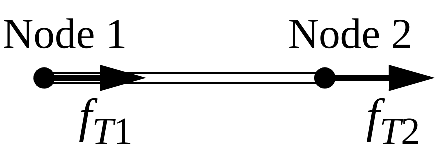
$$\varepsilon_0 = \alpha\Delta T = \alpha\mathbf{N}\Delta\mathbf{T}_e \quad\Rightarrow\quad \mathbf{f}_T = \int_{x_1}^{x_2}\mathbf{B}^T EA\varepsilon_0 \mathrm{d}x = \int_{x_1}^{x_2}\mathbf{B}^T EA\alpha\mathbf{N}\Delta\mathbf{T}_e \mathrm{d}x$$

E.g. use a linear element (natural coordinate: $0 \le \xi \le 1$)
and assume that $EA\alpha$ is constant, then

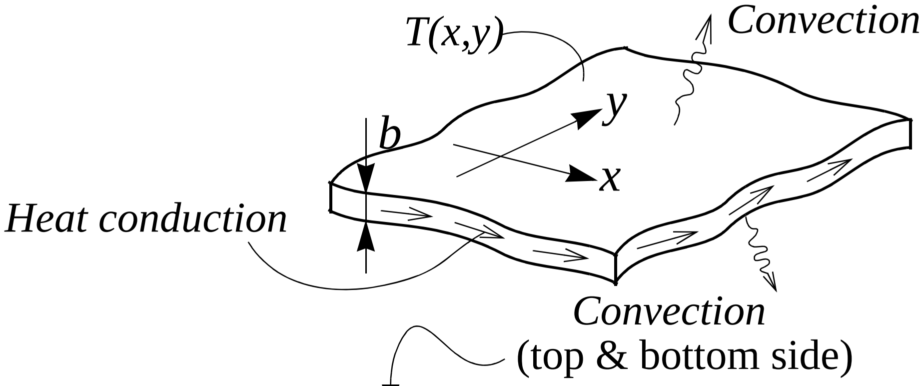
$$\mathbf{N} = \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \quad\Rightarrow\quad \mathbf{B} = \frac{1}{l_e}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\mathbf{f}_T = \int_0^1 \mathbf{B}^T EA\alpha\mathbf{N}\Delta\mathbf{T}_e l_e \mathrm{d}\xi = EA\alpha l_e \int_0^1 \frac{1}{l_e}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1-\xi & \xi \end{bmatrix}\begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix}\mathrm{d}\xi$$

$$= EA\alpha\frac{1}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix} = EA\alpha\underbrace{\frac{\Delta T_2 + \Delta T_1}{2}}_{\textit{mean value in the element!}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_{T1} \\ f_{T2} \end{bmatrix}$$



FEM for heat transfer problems in 2D



Strong Form:

$$\frac{\partial}{\partial x}\left(k_x b\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y b\frac{\partial T}{\partial y}\right)+qb-2h(T-T_\infty) = 0$$

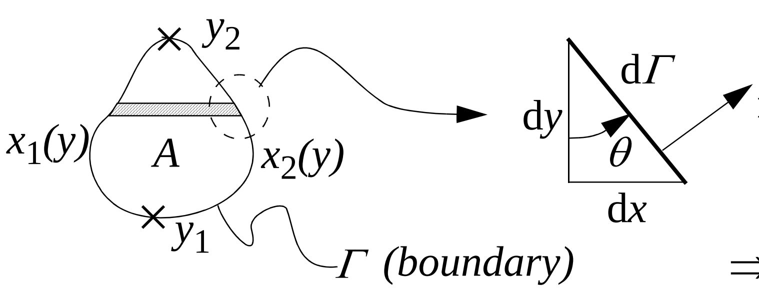
Weak Form:

1. Weighted residual on integral form

$$\int_A v(x,y)\left(\frac{\partial}{\partial x}\left(k_x b\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y b\frac{\partial T}{\partial y}\right)+qb-2h(T-T_\infty)\right)\mathrm{d}A = 0$$

2. Integrate term 1 & 2 by parts

x-direction: $$\underbrace{\int_A \frac{\partial}{\partial x}\left(v\ k_x b\frac{\partial T}{\partial x}\right)\mathrm{d}A}_{\text{use Gauss' theorem and rewrite}} = \int_A \frac{\partial v}{\partial x}\ k_x b\frac{\partial T}{\partial x}\mathrm{d}A+\underbrace{\int_A v\ \frac{\partial}{\partial x}\left(k_x b\frac{\partial T}{\partial x}\right)\mathrm{d}A}_{\text{term 1}}$$



$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\Rightarrow \mathrm{d}y = \mathrm{d}\Gamma\cos\theta = n_x\mathrm{d}\Gamma$$

$$\int_A \frac{\partial}{\partial x}F\mathrm{d}x\mathrm{d}y = \int_{y_1}^{y_2}\left[\int_{x_1(y)}^{x_2(y)}\frac{\partial}{\partial x}F\mathrm{d}x\right]\mathrm{d}y =$$

$$= \int_{y_1}^{y_2}[F(x_2(y),y)-F(x_1(y),y)]\mathrm{d}y = \int_\Gamma F n_x \mathrm{d}\Gamma$$

cont. weak form for heat transfer problems in 2D:

Thus, term 1 can be written as

$$\int_A v\,\frac{\partial}{\partial x}\left(k_x b\frac{\partial T}{\partial x}\right)\mathrm{d}A = -\int_A \frac{\partial v}{\partial x}\,k_x\frac{\partial T}{\partial x}b\,\mathrm{d}A + \int_\Gamma v\,k_x\frac{\partial T}{\partial x}n_x b\,\mathrm{d}\Gamma$$

Similarly, term 2 can be written as

$$\int_A v\,\frac{\partial}{\partial y}\left(k_y b\frac{\partial T}{\partial y}\right)\mathrm{d}A = -\int_A \frac{\partial v}{\partial y}\,k_y\frac{\partial T}{\partial y}b\,\mathrm{d}A + \int_\Gamma v\,k_y\frac{\partial T}{\partial x}n_y b\,\mathrm{d}\Gamma$$

Inserted into weak form gives (let $\frac{\partial}{\partial x}(\ ) = (\ )_{,x}$ etc.)

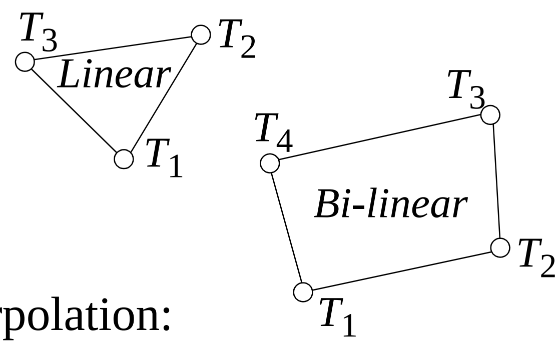
L.h.s.: $$\int_A \underbrace{(v_{,x}k_xT_{,x} + v_{,y}k_xT_{,y})}\,b\,\mathrm{d}A + \int_A v2hT\,\mathrm{d}A$$

$$\begin{bmatrix} v_{,x} & v_{,y}\end{bmatrix}\underbrace{\begin{bmatrix} k_x & 0 \\ 0 & k_y\end{bmatrix}}_{\mathbf{D}}\begin{bmatrix} T_{,x} \\ T_{,y}\end{bmatrix}$$

R.h.s.: $$\int_\Gamma v\,(k_xT_{,x}n_x + k_yT_{,y})\,b\,\mathrm{d}\Gamma + \int_A vqb\,\mathrm{d}A + \int_A v2hT_\infty\,\mathrm{d}A$$

Divide the solid into elements; use the same shape function based interpolation for both temperature and weight function.

E.g. standard linear element:



n_d = number of nodes in the element

*Note! only **ONE** D.O.F. per node!*

Temperature interpolation:

$$T(x, y) = N_1T_1 + \ldots + N_{n_d}T_{n_d} = \begin{bmatrix} N_1 & \ldots & N_{n_d}\end{bmatrix}\begin{bmatrix} T_1 \\ \vdots \\ T_{n_d}\end{bmatrix} = \mathbf{NT}_e$$

Temperature gradient:

$$\begin{bmatrix} T_{,x} \\ T_{,y}\end{bmatrix} = \begin{bmatrix} N_{1,x} & \ldots & N_{n_d,x} \\ N_{1,y} & \ldots & N_{n_d,y}\end{bmatrix}\begin{bmatrix} T_1 \\ \vdots \\ T_{n_d}\end{bmatrix} = \mathbf{BT}_e$$

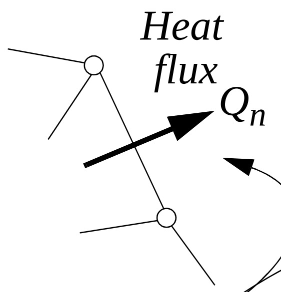
"arbitrary vector"

Weight function: $v(x, y) = \boldsymbol{\beta}^T\mathbf{N}^T \quad\Rightarrow\quad \begin{bmatrix} v_{,x} & v_{,y}\end{bmatrix} = \boldsymbol{\beta}^T\mathbf{B}^T$

FEM Equation for a 2*D* element becomes:

$$\left[\underbrace{\int_{A_e} \overset{n_d\times 2}{\mathbf{B}^T}\overset{2\times 2}{\mathbf{D}}\overset{2\times n_d}{\mathbf{B}}\,b\,\mathrm{d}A}_{\mathbf{K}_{hc}\;(n_d\times n_d)} + \underbrace{\int_{A_e}\mathbf{N}^T 2h\mathbf{N}\,\mathrm{d}A}_{\mathbf{K}_c\;(n_d\times n_d)}\right]\mathbf{T}_e =$$

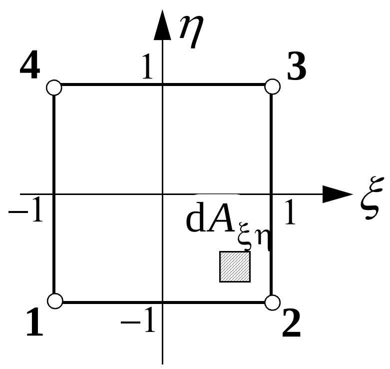
$$= \underbrace{\int_{\Gamma_e}\mathbf{N}^T(-Q_n)b\,\mathrm{d}\Gamma}_{\text{Heat flow across element boundary}} + \underbrace{\overbrace{\int_{A_e}\mathbf{N}^T q b\,\mathrm{d}A}^{\text{Heat supply}} + \overbrace{\int_{A_e}\mathbf{N}^T 2hT_\infty\,\mathrm{d}A}^{\text{Convection term}}}_{\mathbf{f}_b}$$



where $Q_n = \begin{bmatrix} -T_{,x} & -T_{,y}\end{bmatrix}\begin{bmatrix} k_x & 0 \\ 0 & k_y\end{bmatrix}\begin{bmatrix} n_x \\ n_y\end{bmatrix}$

Only external boundaries contributes, since element sharing the same side cancel each others contributions (see text book pp. 309)

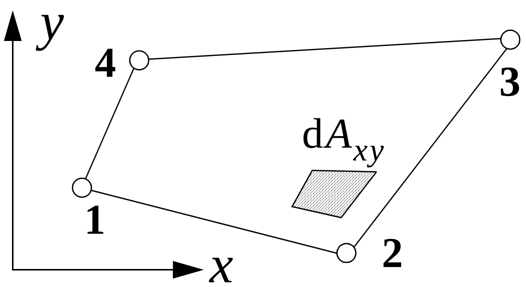
Example: Bi-linear 4-node element (assume that $k_x = k_y = k$, h and b are constants):



$\Rightarrow$

Area relation:

$$|\mathbf{J}|\mathrm{d}A_{\xi\eta} = \mathrm{d}A_{xy}$$



Element matrices/vectors:

Heat conduction:

$$\mathbf{K}_{hc} = kb\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^T\mathbf{B}|\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta$$

Convection:

$$\mathbf{K}_c = 2h\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T\mathbf{N}|\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta$$

$$\mathbf{f}_b = b\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T q|\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta + 2hT_\infty\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T|\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta$$